A COLLECTION OF PROBLEMS

IN ILLUSTRATION OF THE PRINCIPLES OF

THEORETICAL HYDROSTATICS

AND

HYDRODYNAMICS.

BY WILLIAM WALTON, M.A.

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Διὰ τί ἐν τῇ θαλάττῃ μᾶλλον νεῖν δύνανται ἢ ἐν τοῖς ποταμοῖς;
Ἀριστοτέλης.

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PREFACE.

THE success which has within the last few years attended the publication of systematic collections of Examples in several departments of Natural Philosophy and Mathematics, has led me to entertain a belief that a like treatise on the mathematical doctrine of Fluids, would, if composed with a due reference to the necessities of students, be not without its utility. Having been accordingly induced to enter upon this work, I have proposed to myself to furnish the beginner with classes of Problems methodically arranged in elucidation of the various hydrostatical and hydrodynamical theorems ordinarily falling within the province of Academic study. In the fulfilment of this design I have endeavoured, as much as possible, to give to each branch of the subject a proportionate amount of illustration, in order that students, whether of higher or lower mathematical attainment, may be able to meet with a sufficient body of matter applicable to the condition of their knowledge. In carrying out this, my primary object, I have omitted no opportunity of introducing incidentally to the notice of the reader, by historical references, those remarkable memoirs and works of mathematical philosophers in which the first principles of the science of fluids and their most striking consequences were originally unfolded. This secondary purpose of my treatise I have been anxious to fulfil adequately, not only from a wish to enable the higher order of students to acquire more thorough information on particular questions than could have been communicated in accordance with my general design, but also from a conviction that an

acquaintance with the researches of inventive writers in their original form tends greatly to invigorate our conceptions of the fundamental principles of science, and that our interest in its discoveries is ordinarily much augmented by an historical knowledge of their progressive development.

The obligations to various authors under which I have been placed in preparing this volume for the press have been specially acknowledged in the course of the work, whenever I have had any reason to suppose that the source of my information was original. I may add also generally that I have derived great assistance in my undertaking from various examination papers which have been published from time to time in the University of Cambridge. The solutions of those problems which have been extracted from the works of the earlier mathematicians, as well as many for which I am indebted to more modern writers, are almost all presented in this treatise under an entirely new form.

WILLIAM WALTON.

CAMBRIDGE, *September* 24, 1847.

N.B.—A table of Errata, should one be found requisite, is intended to be published in November 1848.

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HYDROSTATICS.

CHAPTER I.

NORMAL PRESSURE OF FLUIDS.

SECTION I.

Normal Pressure of Homogeneous Incompressible Fluids on the Surfaces of Immersed Solids.

LET K denote any indefinitely small element of the surface of a solid immersed in a fluid, and let x represent the depth of K below the surface of the fluid; then, g denoting the accelerating force of gravity and ρ the density of the fluid, $g\rho Kx$ will represent the pressure of the fluid on K, and the magnitude of the pressure on the whole surface of the solid will be equal to

$$g\rho\Sigma\,(Kx),$$

where Σ denotes the summation of a series of terms, of which Kx is the general type, the limits of the summation being defined by the boundary of the immersed surface.

COR. 1. If $\bar{x}$ denote the depth of the centre of gravity of the surface of the solid below the surface of the fluid, then, A representing the surface of the solid, we know that

$$\Sigma\,(Kx) = A\bar{x},$$

and therefore the pressure on A is equal to

$$g\rho A\bar{x},$$

that is, to the weight of a column of the fluid the base of which is equal to the surface of the solid immersed and the height to the depth of the centre of gravity of the surface of the solid

below the surface of the fluid. This proposition was first enunciated generally by Cotes, in his *Hydrostatical and Pneumatical Lectures*, p. 37, 3rd edit.*

When the magnitude of the area (A) and the position of its centre of gravity are known, this corollary enables us at once to determine the whole pressure upon it.

Cor. 2. Suppose that the area A is divisible into a finite number of areas of known magnitudes, $A_{,}$, $A_{,,}$, $A_{,,,}$, &c., the positions of the centres of gravity of which are known, then the pressure on (A) will be equal to

$$g\rho\,(A_{,}\,\bar{x}_{,} + A_{,,}\,\bar{x}_{,,} + A_{,,,}\,\bar{x}_{,,,} + \ldots),$$

where $\bar{x}_{,}, \bar{x}_{,,}, \bar{x}_{,,,}, \ldots$ denote the depths of the centres of gravity of $A_{,}$, $A_{,,}$, $A_{,,,}, \ldots$ respectively below the surface of the fluid.

Whenever, as is most commonly the case, the position of the centre of gravity of the surface of the immersed solid is not known, and it cannot be subdivided into any finite number of areas of which the centres of gravity are known, the evaluation of the expression $g\rho\Sigma\,(Kx)$ must be effected by means of the Integral Calculus.

We have supposed above that the surface of the fluid is free from pressure: should this not be the case, we must consider x to denote the sum of the depth of K below the surface of the fluid and the length (H) of a vertical column of the fluid which would exert a pressure on its base equal to that exerted upon an equal area in the surface of the fluid. This amounts to supposing the depth at any point of the fluid to be $x + H$ instead of x.

The principles of the determination of the pressure of fluids, on plane surfaces, were first laid down by Stevin. His hydrostatical investigations may be seen in the third volume of his *Hypomnemata Mathematica*, translated into Latin from the Dutch by Snell and published at Leyden in the year 1608, or in the fourth volume of *Les Œuvres Mathématiques* de Simon Stevin de Bruges, par Albert Girard Samielois, Mathematicien,

* The first edition of these Lectures was published in 1737, by Dr. Smith, Master of Trinity College, Cambridge.

published at Leyden in the year 1634. In this work the Hydrostatic Paradox, the discovery of which is due to Stevin, or the proposition that a fluid of any assigned weight may be so employed as to produce any assigned pressure, either upwards or downwards, is explained and exemplified. In the fifth book of this treatise, Stevin illustrates his conceptions of fluid action by the discussion of the following question, which, as having relation to the fundamental principles of the transmission of hydrostatical pressure, and as serving to exemplify the accuracy of his mechanical ideas of fluids, we will here quote at length.

"*Declarer la raison pourquoy un homme nageant au fond de l'eau, ne meurt pour la grande quantité d'eau, qui est au dessus de luy.*

"Soit un homme 20 pieds de profondeur dans l'eau, le pied d'eau pesant 65lb, et la superfice entière de son corps 10 pieds: cela estant ainsi, 13000lb presseront contre son corps. Partant on pourroit demander, comment il est possible qu'une personne ne creve d'une si grande charge? A quoy la response sera telle.

"A. Tout pressement qui blesse le corps, pousse quelque partie du corps hors de son lieu naturel.

"O. Ce pressement causé par l'eau, ne pousse aucune partie du corps hors de son lieu naturel.

"O. Ce pressement donc causé par l'eau, ne blesse nullement le corps.

"La mineure est manifeste par l'experience, dont la raison est, que s'il y avoit quelque chose qui soit poussée hors de son lieu, il faudroit que cela rentrast en un autre lieu, mais ce lieu n'est pas dehors, à cause que l'eau presse de tout costé egalement (quant à la partie de dessous, elle est un peu plus pressée que celle de dessus, ce qui n'est d'aucune estime, d' autant que telle différence ne peut pousser aucune partie hors de son lieu naturel,) ce lieu n'est pas aussi dedans le corps, car il n'y a rien de vuide non plus que dehors; d'où il s'ensuit que les parties s'entrepoussent egalement, pource que l'eau a une mesme raison à l'entour du corps. Ce lieu-là donc n'est dehors, ny dedans le corps, et par consequent en nulle part, ce qui fait que nulle partie n'est poussée hors de son lieu, et partant ne blesse nullement le corps.

"Ce que pour declarer plus apertement, soit *ABCD* (fig. 1) une eau, ayant au fond *DC* un trou, fermé d'une broche *E*, sur lequel fond gist un homme *F*, ayant son dos sur *E*; ce qu' estant ainsi, l'eau le pressant de tout costé, celle qui est dessus luy ne pousse aucune partie hors de son lieu.

Mais si on veut voir par effect que cecy est la cause véritable ; il ne faut qu' oster la broche *E*; alors il n'y aura aucun poussement contre son dos en *E*, comme aux autres lieux de son corps; pourtant aussi son corps patira là une compression, voire aussi forte; assavoir autant que pèse la colomne d'eau, ayant le trou *E* pour base, et *AD* hauteur; et ainsi le dessein est demonstré apertement."

1. A rectangular parallelogram is immersed in a fluid; to find the whole pressure upon it.

Let a, b, be the lengths of two unequal sides, and h the depth of its centre of gravity, that is, of its middle point, below the surface of the fluid; then, by Cor. (1), the whole pressure upon it will be equal to

$$g\rho \, ab \, h.$$

This result shews that, so long as h remains constant, the pressure on the parallelogram will remain the same at whatever inclination it may be placed to the horizontal surface of the fluid.

2. A solid cylinder is immersed in a fluid, the depths of the centres of its circular ends being h, h', its radius r, and its length l; to determine the pressure on its surface, including its extremities.

The areas of its two ends are each equal to πr^2, and, the depth of the centre of gravity of one being h, and of the other h', the pressures on the two ends are

$$g\rho.\pi r^2 h, \quad g\rho.\pi r^2 h'.$$

Again, the area of the convex portion of the cylinder being $2\pi rl$ and the depth of its centre of gravity $\frac{1}{2}(h + h')$, the pressure upon it will be

$$g\rho \,.\, 2\pi rl \,.\, \tfrac{1}{2}(h + h').$$

Hence the whole pressure on the cylinder will be equal to

$$\pi g\rho r.\{r(h+h')+l(h+h')\}$$
$$=\pi g\rho r(r+l)(h+h').$$

This result shews that, so long as $\frac{1}{2}(h+h')$, the depth of the centre of gravity of the whole cylinder, remains constant, the whole pressure on the cylinder will be invariable, however the inclination of its axis to the surface of the fluid may be varied.

We have solved this problem by means of Cor. (2). It might have been solved however somewhat more simply by means of Cor. (1). In fact, the whole surface of the cylinder being equal to

$$2\pi r^2+2\pi rl=2\pi r(r+l),$$

and the depth of the centre of gravity of the surface, which is evidently the middle point of the axis, being equal to $\frac{1}{2}(h+h')$, it follows that the whole pressure on the cylinder must be equal to

$$\pi g\rho r(r+l)(h+h').$$

3. An isosceles triangle is immersed vertically in a fluid with its vertex coincident with the surface of the fluid and its base horizontal; to determine how it must be divided by a line parallel to the base, that the pressures upon the upper and lower portions may be respectively in the ratio of $m:n$.

Let h be the altitude of the proposed triangle, and h' that of the triangle cut off by the dividing line. Let c be the length of the base of the proposed triangle; that of the triangle cut off will be equal to $c.\dfrac{h'}{h}$.

The pressures on these two triangles will be

$$g\rho.\tfrac{2}{3}h.\tfrac{1}{2}ch=\tfrac{1}{3}g\rho h^2c,$$

$$\text{and}\quad g\rho.\tfrac{2}{3}h'.\tfrac{1}{2}c\frac{h'}{h}.h'=\tfrac{1}{3}g\rho\frac{h'^3c}{h}.$$

The pressure on the lower portion of the proposed triangle will therefore be

$$\tfrac{1}{3}g\rho.\frac{c}{h}.(h^3-h'^3).$$

Hence, by the hypothesis,

$$h'^3 : h^3 - h'^3 :: m : n,$$

$$h'^3 : h^3 :: m : m + n,$$

$$h' = h\left(\frac{m}{m+n}\right)^{\frac{1}{3}}.$$

COR. If $h' = \frac{1}{2}h$, then $\frac{1}{2} = \left(\frac{m}{m+n}\right)^{\frac{1}{3}}$,

and therefore $$\frac{m}{n} = \frac{1}{7};$$

or, if the dividing line bisect the perpendicular altitude, the pressure on the lower portion of the triangle will be seven times as great as that on the upper.

4. To find the pressure which a diver sustains when the centre of gravity of the surface of his body is 32 feet under water.

"The surface of a middle-sized human body is about 10 square feet. Multiply then 32, the depth of the center under water, by 10, the surface of the body, and the product, or 32 times 10 solid feet, will be a magnitude of water whose weight is equivalent to the pressure which the diver sustains. A cubick foot of water has been found by experiment to weigh 1000 averdupois ounces, therefore 32 times 10 feet, or 16 times 20 feet of water, will weigh 16 times 20000 averdupois ounces, or 20000 averdupois pounds. This therefore is the pressure of the water to which a diver at 32 feet depth is exposed."—Cotes: *Hydrostatical and Pneumatical Lectures*, Lect. 3.

5. To compare the pressures on the upper and lower portions of a hemispherical vessel full of fluid, the axis of the vessel being vertical, and the two portions being separated by a horizontal plane bisecting the axis.

Let θ be the inclination of any radius of the bowl to its axis; then, r representing the radius, the pressure on an annular strip of the bowl included between two consecutive horizontal planes, will be equal to

$$g\rho \,.\, 2\pi r \sin\theta \,.\, r d\theta \,.\, r\cos\theta = \pi g\rho r^3 \sin 2\theta\, d\theta.$$

Hence the required ratio will be equal to

$$\frac{\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \sin 2\theta d\theta}{\int_0^{\frac{1}{3}\pi} \sin 2\theta d\theta} = \frac{\frac{1}{2}(1-\frac{1}{2})}{\frac{1}{2}(1+\frac{1}{2})} = \frac{1}{3}.$$

6. A plane surface, bounded by the arc of a parabola and the tangents at its extremities, is immersed in water, the point of intersection of the tangents coinciding with the surface of the fluid, and the area of the parabola being vertical: to determine the whole pressure on the plane surface.

Let OA, OB, (fig. 2) be the two tangents to the parabola $HABK$, and let OA, OB, produced indefinitely, be the axes of x, y, respectively. Let α, β, be the inclinations of OA, OB, respectively, to the horizon.

The area of an elementary parallelogram Pp, the coordinates of P, p, being x, y, and $x + dx$, $y + dy$, respectively, will be

$$dx\,dy \sin(\alpha + \beta),$$

and the depth of Pp below the surface of the fluid will be equal to

$$x \sin \alpha + y \sin \beta.$$

Hence the pressure on the area AOB will be equal to

$$g\rho \sin(\alpha + \beta) \iint dx\,dy\,(x \sin \alpha + y \sin \beta).$$

Now, if $OA = a$, $OB = b$, the equation to the parabola will be

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1;$$

hence, integrating with regard to y from $y = 0$ to $y = y$, the y in the limit being the ordinate of a point in the curve, and then from $x = 0$ to $x = a$, we have

$$\begin{aligned} \iint x\,dx\,dy &= \int x\,dx\,.\,y \\ &= b\int x\,dx\,.\left\{1 - \left(\frac{x}{a}\right)^{\frac{1}{2}}\right\}^2 \\ &= b\int x\,dx\left\{1 - 2\left(\frac{x}{a}\right)^{\frac{1}{2}} + \frac{x}{a}\right\} \\ &= b\left(\tfrac{1}{2}a^2 - \tfrac{4}{5}a^2 + \tfrac{1}{3}a^2\right) = \tfrac{1}{30}a^2 b. \end{aligned}$$

Similarly we have $\iint y\,dx\,dy = \frac{1}{30}b^3 a.$

Hence the pressure on the area AOB is equal to

$$\tfrac{1}{30}g\rho ab \sin(\alpha+\beta)\,\{a\sin\alpha + b\sin\beta\}.$$

7. To determine the pressure on a loop of the Lemniscata of James Bernoulli, the axis of the loop being vertical, and its vertex just touching the surface of the fluid.

The polar equation to the curve is

$$r^2 = a^2\cos 2\theta,$$

the axis of the loop being the prime radius vector, and the vertex being the pole.

Hence, the depth of an elementary polar area $r d\theta dr$ below the surface of the fluid being $r\cos\theta$, the pressure on the loop will be equal to

$$\begin{aligned}
&g\rho \iint r\,d\theta\,dr\,.\,r\cos\theta\\
&= 2g\rho\int_0^r\int_0^{\frac{1}{4}\pi} r^2\cos\theta\,d\theta\,dr\\
&= \tfrac{2}{3}g\rho\int_0^{\frac{1}{4}\pi} r^3\cos\theta\,d\theta\\
&= \tfrac{2}{3}g\rho\,a^3\int_0^{\frac{1}{4}\pi}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\,d\theta\\
&= \tfrac{2}{3}g\rho\,a^3\int_0^{\frac{1}{\sqrt{2}}}(1-2\sin^2\theta)^{\frac{3}{2}}\,d\sin\theta.
\end{aligned}$$

Put $2\sin^2\theta = \sin^2\phi$, or $\sin\theta = \dfrac{1}{\sqrt{2}}\sin\phi$; then we have for the pressure

$$\begin{aligned}
&\tfrac{2}{3}g\rho\,a^3\,\frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi}\cos^4\phi\,d\phi\\
&= \frac{2}{3\sqrt{2}}\,.\,g\rho\,a^3.\,\tfrac{1}{8}\int_0^{\frac{1}{2}\pi}(\cos 4\phi + 4\cos 2\phi + 3)\,d\phi\\
&= \frac{1}{12\sqrt{2}}\,.\,g\rho\,a^3.\,\frac{3\pi}{2} = \frac{g\rho\pi a^3}{8\sqrt{2}}.
\end{aligned}$$

8. A hemisphere, with a flat lid, as nearly as possible filled with fluid, is held with a point in its edge uppermost; to find its position when the sum of the pressures on the curve and the plane surfaces is the greatest possible.

Let θ be the inclination of the axis of the hemisphere to the vertical, and let r denote the radius.

Then the area of the base is equal to πr^2, and that of the curve surface to $2\pi r^2$. Also the depth of the centre of gravity of the base below the highest point of the rim is equal to $r \sin \theta$, and the depth of the centre of gravity of the curve surface, which is at the middle point of the axis of the hemisphere, below the same point, is equal to

$$r \sin \theta + \tfrac{1}{2} r \cos \theta.$$

Hence the whole pressure on the two surfaces of the hemisphere is equal to

$$g\rho \left\{\pi r^2 . r \sin \theta + 2\pi r^2 . (r \sin \theta + \tfrac{1}{2} r \cos \theta)\right\}$$
$$= \pi g \rho r^3 (3 \sin \theta + \cos \theta).$$

When this expression is a maximum,

$$3 \cos \theta - \sin \theta = 0,$$

or
$$\tan \theta = 3,$$

which determines the required position.

9. A rectangular board is immersed vertically in a fluid, one side of the board being coincident with the surface; to find the pressure on the board.

If a denote the length of a horizontal and b of a vertical side of the board, and ρ the density of the fluid, the required pressure will be equal to
$$\tfrac{1}{2} g \rho a b^2.$$

Bossut: *Traité d'Hydrodynamique*, tom I. p. 33.

10. A triangle of any form is immersed vertically in a fluid with one side in the surface; to determine the whole pressure on the triangle.

Let c represent the length of the side in the surface of the fluid, and h the distance of the opposite angle from this side; then the required pressure will be equal to

$$\tfrac{1}{6} g \rho c h^2.$$

11. A board in the shape of an isosceles right-angled triangle with the squares on its sides, is placed with the upper side of the square opposite to the right angle in the surface of the fluid;

to compare the sum of the pressures on the squares containing the right angle with the pressure on the square opposite to the right angle.

If P = the pressure on the square opposite to the right angle, and Q = the sum of the pressures on the squares containing the right angle,
$$Q = 3P.$$

12. A rectangle is immersed vertically in a fluid with one angle in the surface of the fluid; to find the pressure upon it, and to determine the inclination of its sides to the surface when the pressure is a maximum.

If $2a$, $2b$, be two of its unequal sides, and θ the inclination to the horizon of the diagonal through the angle which is in the surface of the fluid, the pressure will be equal to
$$4g\rho ab\,(a^2+b^2)^{\frac{1}{2}}.\,\sin\theta,$$
which will evidently be the greatest possible when $\theta = \frac{1}{2}\pi$, that is, when its sides $2a$, $2b$, are inclined to the surface at angles $\tan^{-1}\dfrac{a}{b}$, $\tan^{-1}\dfrac{b}{a}$, respectively.

13. $ABCD$ is a parallelogram, the diagonals AC, BD, of which intersect in E, AB being in the surface of the fluid: to compare the pressures on the three triangles AEB, BEC, CED.

If P, Q, R, represent the three pressures,
$$P : Q : R :: 1 : 3 : 5.$$

14. A triangle, of which the area is A, is immersed in a fluid, the angular points being at depths h, h', h'', below the surface of the fluid; to find the pressure on the triangle.

The pressure is equal to
$$\tfrac{1}{3}g\rho A\,(h+h'+h'').$$

15. A rod, inclined at any angle to the horizon, is just immersed in a fluid; to divide it into four parts, which shall be equally pressed.

Let a represent the length of the rod, and x, x', x'', x''', the lengths of the four parts taken in order, beginning with the highest. Then
$$x=\tfrac{1}{2}a,\quad x'=\tfrac{1}{2}a\,(\sqrt{2}-1),\quad x''=\tfrac{1}{2}a\,(\sqrt{3}-\sqrt{2}),\quad x'''=\tfrac{1}{2}a\,(2-\sqrt{3}).$$

16. A cubical vessel, filled with fluid, is held so that one of its diagonals is vertical; to compare the pressures on one of its higher and one of its lower faces.

If P be the pressure on a higher and Q on a lower face, then

$$Q = P(2\sqrt{3} - 1).$$

17. A symmetrical pyramid is immersed in fluid with the surface of which its vertex just coincides; the axis of the pyramid is vertical: to find the pressure on the surface of the pyramid, exclusively of the base, the length of each of the n sides of its polygonal base being a, and the height of each of its triangular faces being h.

The required pressure is equal to

$$\tfrac{1}{3} g\rho nah \left(h^2 - \tfrac{1}{4} a^2 \cot^2 \frac{\pi}{n}\right)^{\frac{1}{2}}.$$

18. An ellipse is placed with its axis vertical and the extremity of its axis major in the surface of a fluid; to compare the pressure on the ellipse with the pressure on the circle of curvature at its highest vertex.

If P denote the former and Q the latter of these pressures, then, $2a$, $2b$, being the major and minor axes of the ellipse,

$$P : Q :: a^5 : b^5.$$

19. A circular area is immersed vertically in a fluid, and has a point of its circumference in the surface; the circular area being divided into two parts by a straight line drawn from its highest point at an angle of 45° to the horizon, to determine the ratio of the pressure on the larger to the pressure on the smaller portion.

The required ratio is equal to

$$\frac{9\pi + 8}{3\pi - 8}.$$

20. A quadrant of a circle ACB is divided into two sectors ACP, PCB, and is immersed in a fluid so that AC is coincident with the surface; to compare the areas of the two sectors when they both experience the same pressure.

The area of the sector ACP must be double of that of the sector PCB.

21. A semicircular area is placed in a fluid with its vertex downwards, and its diameter coincident with the surface; to divide the area into n portions by horizontal ordinates, such that the pressures upon all the portions may be equal.

If a be the radius of the circle, and y_x the lower ordinate of the x^{th} portion, the diameter of the semicircle being the higher ordinate of the first portion, then

$$y_x = \left(\frac{n-x}{n}\right)^{\frac{1}{3}} a,$$

or the values of $y_1, y_2, y_3, \ldots$ will be respectively

$$\left(\frac{n-1}{n}\right)^{\frac{1}{3}} a,\ \left(\frac{n-2}{n}\right)^{\frac{1}{3}} a,\ \left(\frac{n-3}{n}\right)^{\frac{1}{3}} a, \ldots\ldots$$

22. An elliptic area is immersed vertically in fluid, the major axis coinciding with the surface; to find the pressure on the area included between the arc and the line joining the extremities of two conjugate diameters.

If a, b, denote the semi-axes of the ellipse, and α, β, the depths of the extremities of the two conjugate diameters, the required pressure will be equal to

$$\tfrac{1}{6}\pi\rho ab\,(\alpha + \beta).$$

23. A parabola is immersed in a fluid with its axis vertical, its vertex A being at the surface; S being its focus and P a point in its arc, to determine the pressure on the area included between the straight lines SA, SP, and the parabolic arc AP.

If $SA = m$ and $SP = r$, the required pressure will be equal to

$$\tfrac{1}{15}g\rho\,(rm - m^2)^{\frac{1}{2}}\,.\,(2r^2 + rm + 2m^2).$$

If $r = 2m$, or if SP be the semi-latus-rectum, the pressure will be equal to $$\tfrac{4}{3}g\rho m^3.$$

24. A cycloidal area is immersed in a fluid, so that the tangent at its vertex lies in the surface; to compare the pressure

on the whole area of the cycloid with that on a circular area of which the axis of the cycloid is a diameter.

If P denote the pressure on the cycloidal and Q on the circular area, then

$$P : Q :: 7 : 2.$$

25. An elliptic area is placed with its major axis vertical, and its vertex is the surface of a fluid; to compare the pressure of the fluid on the area included by the evolute of the ellipse with the pressure on the whole elliptic area.

If e be the eccentricity of the ellipse, the ratio of the pressure on the area of the evolute to that on the area of the ellipse is equal to

$$\frac{3}{8} \cdot \frac{e^4}{1 - e^2}.$$

26. To find the whole pressure on the surface of a solid cone, including its circular base, when it is immersed in a fluid with its axis vertical and its vertex just at the surface.

If r be the radius of the base, h the length of the axis, and l the distance of each point in the periphery of the base from the vertex, the required pressure will be equal to

$$\tfrac{1}{3}\pi g\rho h r\,(2l + 3r).$$

27. A solid hemisphere is immersed in a fluid with its axis inclined at an angle θ to the vertical, the surface of the fluid being a tangent plane to the hemisphere; to find the whole pressure on the convex surface of the hemisphere.

If a denote the radius of the hemisphere, the required pressure will be equal to

$$\pi g\rho a^3\,(2 - \cos\theta).$$

28. To divide a hollow sphere just filled with fluid by a circle parallel to the horizon into two parts which shall be equally pressed.

Let θ represent the inclination of a radius of the sphere passing through the circumference of the required circle to a line drawn vertically upwards from the centre of the sphere; then

$$\cos\theta = 1 - \sqrt{2}.$$

29. A hollow segment of a sphere rests on its base on an inclined plane; supposing it to be just filled with fluid, to find the pressure on the spherical surface.

If α denote the inclination of the plane to the horizon, β the angle subtended at the centre of the sphere by a diameter of the base of the segment, and r the radius of the sphere, the required pressure will be equal to

$$4\pi g\rho r^3 \sin^2 \frac{\beta}{4} \left(1 - \cos \alpha \cos^2 \frac{\beta}{4}\right).$$

30. The axis of a given hollow cone filled with fluid is inclined at a given angle to the horizon; to find how much of the fluid will flow out, and to determine the pressure exerted by the remainder upon the conical surface.

Let 2α = the vertical angle of the cone, β = the inclination of its axis to the horizon, a = the radius of its base; then the volume of the fluid discharged will be equal to

$$\tfrac{1}{3}\pi a^3 \cot \alpha \,.\, \left\{\frac{\sin^{\frac{3}{2}} (\beta + \alpha) - \sin^{\frac{3}{2}} (\beta - \alpha)}{\sin^{\frac{3}{2}} (\beta + \alpha)}\right\},$$

and the required pressure will be equal to

$$\tfrac{1}{3}\pi g\rho a^3 . \frac{\cos \alpha \,.\, \sin \beta}{\sin^2 \alpha} \,.\, \left\{\frac{\sin (\beta - \alpha)}{\sin (\beta + \alpha)}\right\}^{\frac{3}{2}}.$$

SECTION II.

Normal Pressure of Heterogeneous Incompressible Fluids on the Surfaces of Immersed Solids.

Let K denote any indefinitely small element of the surface of a solid immersed in a heterogeneous fluid, and let p denote the unit of pressure of the fluid at the area K; then pK will represent the pressure of the fluid on K, and the magnitude of the pressure on the whole surface of the solid will be equal to

$$\Sigma\,(Kp),$$

where Σ denotes the summation of a series of terms, of which Kp is the general type, the limits of the summation being defined by the boundary of the immersed surface. The value of p will be given by the formula

$$p = \Sigma (g\rho\, dx) = g\Sigma (\rho\, dx),$$

ρ denoting the density of the fluid at a depth x below the surface of the fluid, the summation being performed with respect to x from $x = 0$ to $x = x$, the value of x in the second limit being the depth of K below the surface.

Supposing ρ to be invariable, then

$$p = g\Sigma (\rho\, dx) = g\rho\, \Sigma (dx) = g\rho x,$$

and $$\Sigma (Kp) = g\rho\, \Sigma (Kx),$$

and the formulæ here given will degenerate into those of the preceding section for the pressure of homogeneous fluids.

If ρ be not continuous, we must divide the depth of K below the surface into a series of parts, such that ρ may be continuous for each part, and must then take the sum of the values of $g\Sigma (\rho\, dx)$ for the several strata as constituting the value of p.

1. A cylinder, the axis of which is vertical, is filled with fluid, the density of which varies directly as the depth; to find the whole pressure on the concave surface of the cylinder.

Let h denote the length of the cylinder, c the circumference of a circular section, ρ the density at the lowest points of the fluid and ρ' at any depth x. Then

$$dp = g\rho' dx = \frac{g\rho}{h}\, x\, dx,$$

$$p = \frac{g\rho}{h} \int_0^h x\, dx = \frac{g\rho x^2}{2h}.$$

Hence the whole pressure required is equal to

$$\int_0^h \frac{g\rho x^2}{2h} \cdot c\, dx = \tfrac{1}{6} g\rho c h^2.$$

Encycl. Metrop. Mixed Sciences, vol. I. p. 176.

2. A circular area is immersed vertically in fluid, its highest point just touching the surface of the fluid; to determine the

whole pressure on the area, the density of the fluid varying directly as the depth.

Let β represent the density of the fluid at the centre of the circle; the density, at a depth x below the surface, will therefore be $\frac{\beta x}{a}$, where a is the radius of the circle. Then

$$dp = g\rho\, dx = \frac{g\beta}{a}\, x\, dx,$$

$$p = \frac{g\beta}{2a}\, x^2.$$

Let r denote the distance of any point of the circular area from its centre, and θ the inclination of this distance to the vertical. Then, the pressure on an elemental area $rd\theta\, dr$ being

$$rd\theta\, dr . p = \frac{g\beta}{2a}\, rd\theta\, dr . x^2$$

$$= \frac{g\beta}{2a}\, rd\theta\, dr . (a - r\cos\theta)^2,$$

the whole pressure on the circular area will be equal to

$$\frac{g\beta}{2a}\int_0^a\int_0^{2\pi} r(a^2 - 2ar\cos\theta + r^2\cos^2\theta)\, d\theta\, dr$$

$$= \tfrac{1}{2}ga^3\beta\int_0^{2\pi}(\tfrac{1}{2} - \tfrac{2}{3}\cos\theta + \tfrac{1}{4}\cos^2\theta)\, d\theta$$

$$= \tfrac{1}{2}ga^3\beta\,(\pi + \tfrac{1}{4}\pi) = \tfrac{5}{8}\pi g a^3\beta.$$

3. Equal masses of n different fluids, the densities of which, beginning with the highest fluid, are $\rho_1, \rho_2, \rho_3, \ldots\ldots \rho_n$, being placed in a cylindrical vessel the axis of which is vertical, to compare the pressures which they exert upon the side of the vessel.

Let $a_1, a_2, a_3, \ldots a_n$, be the portions of the axis of the cylinder which are occupied by the fluids of densities $\rho_1, \rho_2, \rho_3, \ldots\ldots \rho_n$, respectively. Let z denote the depth of an annular strip $2\pi r dz$ of the cylinder, r being its radius, below the upper surface of the x^{th} fluid. Then, p being the unit of pressure at the annulus,

$$p = g(\rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3 + \ldots + \rho_{x-1} a_{x-1}) + g\rho_x z:$$

but, the masses of the several fluids being equal,

$$\rho_1 a_1 = \rho_2 a_2 = \rho_3 a_3 = \ldots\ldots = \rho_{x-1} a_{x-1};$$

hence $$p = g(x-1)\rho_1 a_1 + g\rho_x z,$$

and therefore, P_x denoting the pressure on the x^{th} portion of the cylinder, we have

$$P_x = \int_0^{a_x} 2\pi r \,.\, dz \,.\, g\,\{(x-1)\rho_1 a_1 + \rho_x z\}$$
$$= 2\pi r \,.\, g\,\{(x-1)\rho_1 a_1 a_x + \tfrac{1}{2}\rho_x a_x^2\};$$

or, since $\rho_x a_x = \rho_1 a_1$,

$$P_x = \pi g r \rho_1 a_1 \{(2x-2) a_x + a_x\}$$
$$= \pi g r \rho_1 a_1 (2x-1) a_x$$
$$= \pi g r \,.\, \rho_1^2 a_1^2 \,.\, \frac{2x-1}{\rho_x}.$$

The expression for P_x shews that the pressures on the first, second, third, &c. portions of the cylinder are proportional to

$$\frac{1}{\rho_1},\ \frac{3}{\rho_2},\ \frac{5}{\rho_3},\ \frac{7}{\rho_4},\ldots\ldots$$

4. To find the whole pressure on the horizontal base of a vessel full of fluid, supposing the density of the fluid to vary as the altitude above the base.

If ρ denote the density at the surface, h the height of the cylinder, and A the area of its base, the required pressure will be equal to $$\tfrac{1}{2}g\rho hA.$$

The same result will express the pressure on the base if the density be supposed to vary as the depth, and ρ to denote the density at the base.

Encycl. Metrop. vol. I. Mixed Sc. p. 176.
Bossut: *Traité d'Hydrodynamique*, p. 39.

5. Masses of n different fluids, which do not mix, the densities of which are $\rho_1, \rho_2, \rho_3, \ldots \rho_n$, being placed in a vessel, to determine the pressure on the base of the vessel, which is horizontal.

If A represent the area of the base, and $h_1, h_2, h_3, \ldots h_n$, the depths of the strata, the required pressure is equal to

$$gA\,(\rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3 + \ldots\ldots + \rho_n h_n).$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 37.

6. A cylinder, the axis of which is vertical, is filled with fluid, the density of which varies directly as the depth; to find the pressure on the concave surface of a portion of the cylinder included between two horizontal sections.

If h denote the length of the cylinder, c the circumference of a circular section, ρ the density at the lowest points of the fluid, and b, b', the depths of the two horizontal sections, the required pressure will be equal to

$$\tfrac{1}{6}g\rho c \, . \, \frac{b'^3 - b^3}{h} \, .$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 40.

7. A semicircle, having its diameter in the surface of a fluid, is divided into three equal sectors; to compare the sum of the pressures on the outer sectors with the pressure on the middle sector, the density of the fluid being supposed to vary as the depth.

If P represent the sum of the pressures on the two outer sectors, and Q the pressure on the middle sector, then

$$P : Q :: 4\pi - 3\sqrt{3} : 2\pi + 3\sqrt{3}.$$

8. A cycloidal area is immersed in fluid, with its axis vertical and its vertex at the surface; to determine the whole pressure on the cycloid, the density of the fluid varying directly as the depth.

If a denote the radius of the generating circle, and β the density of the fluid at the middle point of the axis of the cycloid, the whole pressure on the cycloidal area will be equal to

$$\tfrac{31}{12}\pi a^3 g\beta.$$

9. The density of the fluid being supposed to vary as the depth, to find the pressure upon a triangular plane, one angle of which is a right angle and the base of which coincides with the surface of the fluid, the inclination of the plane to the horizon being given.

Let h be the height of the triangle, l the length of its base, θ the inclination of its area to the horizon, and β the density

of the fluid at a depth a below its surface; then the required pressure will be equal to

$$\frac{g\beta h^3 l}{24a} \cdot \sin^2 \theta.$$

SECTION III.

Normal Pressure of Aeriform Fluids of Uniform Temperature on the Surfaces of Solids.

In elastic or aeriform fluids of uniform temperature, the value of p is to be determined from the two equations

$$dp = g\rho dx, \quad p = k\rho,$$

where k is a constant quantity dependent upon the elasticity of the fluid; the total normal pressure upon the surface of any solid immersed in the fluid being then determined, as in the preceding section, by the evaluation of the expression $\Sigma\,(Kp)$.

Eliminating ρ between the two equations, we have

$$\frac{dp}{p} = \frac{g}{k}\,dx,$$

or, putting $\frac{g}{k} = \lambda$,

$$\frac{dp}{p} = \lambda dx;$$

whence, C being an arbitrary constant, we obtain

$$p = C \,.\, \varepsilon^{\lambda x}.$$

Hence the total normal pressure is equal to

$$C\Sigma\,(K \,.\, \varepsilon^{\lambda x}).$$

In the aeriform fluids of nature, λ is an extremely small quantity, and therefore, approximately, unless x be very large, as for instance in estimating the pressure of the earth's atmosphere at points very near the surface of the earth, we shall have $p = C$, and therefore the formula for the pressure will be reduced to $C\Sigma\,(K)$.

The ancient philosophers were ignorant that air is a ponderable fluid. They conceived all substances as naturally arranging

themselves into two classes, the one class comprehending *heavy* and the other *light* bodies, those bodies being regarded as essentially heavy which tend to fall towards the earth, and those as essentially light which tend to rise from it. This misconception of the mechanical properties of air resulted from their ignorance of the true theory of such phenomena as depending upon the displacement of heavy fluids or bodies by bodies or fluids more heavy, all substances in fact naturally tending to descend. The idea of the ponderable nature of air first presented itself to Galileo; the first confirmation however of this notion by the test of experiment is due to his pupil Torricelli. In accordance with Galileo's idea, the pressure of the atmosphere, as indicated by the Torricellian tube or ordinary barometer, must be greater at the level of the sea than at the summits of mountains. The first independent proof of this conclusion was afforded by Pascal's experiments, in the year 1648, on the mountain of Puy-de-Dome, near Clermont in Auvergne: in ascending from the foot to an altitude of 3000 Paris feet, the barometric column subsided three inches and one eighth of an inch; or, in English measure, in ascending to an altitude of 3204 feet, the height of the quicksilver was diminished by three inches and one third of an inch.

1. An upright cylinder, closed at both ends, is filled with an elastic fluid; to determine the whole pressure on the concave surface of the vessel.

Let β denote the elastic force of the fluid at the upper end of the vessel; then

$$p = C \, . \, \epsilon^{\lambda x}, \quad \beta = C,$$

and therefore

$$p = \beta \, . \, \epsilon^{\lambda x}.$$

Hence, c denoting the circumference of a horizontal section and h the altitude of the cylinder, the required pressure will be equal to

$$\int_0^h c \, . \, dx \, . \, \beta \epsilon^{\lambda x} = c\beta \int_0^h \epsilon^{\lambda x} . \, dx = \frac{c\beta}{\lambda} (\epsilon^{\lambda h} - 1).$$

Supposing λ to be very small and h not very large, we see that, expanding $\epsilon^{\lambda h}$ by powers of λh, and neglecting terms of higher

powers than the first, the expression for the pressure will be reduced to $c\beta h$.

2. If the density of a fluid, supposed to consist of an indefinite number of mutually repellent particles at finite intervals, vary as the pressure, to determine the law of the repulsive force between the particles.

Conceive the fluid to be contained in the cubical space ACE (fig. 3), and then to be reduced by compression into the smaller cubical space ace; and the distances between the particles, occupying a similar position *inter se* in both the spaces, will be as the sides AB, ab, of the cubes, and the densities of the media reciprocally as the containing spaces, viz. the cube of AB and the cube of ab.

In the plane face of the greater cube $ABCD$ take the square DP equal to the plane face db of the smaller cube; then, by the hypothesis,

Pressure of DP on the included fluid
: pressure of db on the included fluid
:: density of fluid DE : density of fluid de
:: $(ab)^3 : (AB)^3$.

But

Pressure of DB on the included fluid
: pressure of DP on the included fluid
:: area DB : area DP
:: $(AB)^2 : (ab)^2$.

Hence

Pressure of DB on the included fluid
: pressure of db on the included fluid
:: $ab : AB$.

Conceive planes FGH, fgh, to be drawn through the two fluids parallel to their faces AE, ae; these planes will divide the fluids into two parts which will press against each other with the same forces with which the fluids are pressed by the planes AC, ac, that is, in the proportion of ab to AB; and accordingly the repulsive forces, by which these pressures are sustained, will be in the same ratio. But, the number of the

particles in the two cubes being the same and their positions being similar, the total mutual pressures of the two portions of the fluids at the planes FGH, fgh, are as the mutual repulsive forces of two similar particles in these two cubes. Hence

The repulsive force between two particles in the larger cube : the repulsive force between two particles similarly situated in the smaller cube :: $ab : AB :: \frac{1}{r} : \frac{1}{r'}$, r and r' being the distances between the two particles in the former and in the latter cube.

Cor. The converse proposition is also true. Suppose, in fact, that the mutual repulsions between two similarly situated particles in the two cubes are as $\frac{1}{r} : \frac{1}{r'}$. Then the total pressures on the faces DB, db, will be as the sums of the forces of the separate particles, or, the number of the particles at these faces being the same, as $\frac{1}{r} : \frac{1}{r'}$, that is, as $ab : AB$. Also

Pressure of DP : pressure of $DB :: (ab)^2 : (AB)^2$;
hence
Pressure of DP : pressure of $db :: (ab)^3 : (AB)^3$
:: density of ACE : density of ace.

Newton: *Principia*, Lib. II. Sect. 5, Prop. 23.

3. A spherical envelope is filled with elastic fluid; to determine the whole pressure on the envelope.

Let β represent the unit of pressure of the fluid at the centre of the envelope, and let a denote its radius; then the required pressure will be equal to

$$\frac{2\pi a\beta}{\lambda} \cdot (\varepsilon^{\lambda a} - \varepsilon^{-\lambda a}).$$

If λ be a very small quantity, then, supposing a not to be large, this expression, by puting $1 + \lambda a$, $1 - \lambda a$, for $\varepsilon^{\lambda a}$, $\varepsilon^{-\lambda a}$ respectively, is reduced to

$$4\pi a^2\beta,$$

a value which might have been obtained at once by multiplying

the whole area of the spherical envelope, viz. $4\pi a^2$, by the invariable unit of pressure of the fluid, viz. β.

4. A fluid being supposed to consist of an indefinite number of particles at finite intervals, and the repulsive force between two neighbouring particles being supposed to vary inversely as the n^{th} power of their distance, to determine the relation between the density and the elastic force of the fluid.

$$\text{Elastic force} \propto (\text{Density})^{\frac{n+2}{3}}.$$

The converse proposition is also true.

Newton: *Principia*, Lib. II., Sect. 5, Prop. 23, *Scholium*.

CHAPTER II.

RESULTANT PRESSURES OF FLUIDS ON THE PLANE SURFACES OF IMMERSED SOLIDS.

SECTION I.

Homogeneous Incompressible Fluids.

LET K be an elemental portion of a plane area immersed in a fluid at any inclination to the horizon, and let h be the depth of K below the surface of the fluid; then, ρ being the density of the fluid, the pressure on K will be equal to $g\rho hK$. On every such small element of the immersed area a corresponding force will act. The immersed area will therefore be acted upon at right angles by an indefinite number of parallel forces all tending in the same direction; let the resultant of these forces be R, and, taking any system of coordinate axes, rectangular or oblique, in the plane of the immersed area, let $\bar{x}, \bar{y}$, be the coordinates of the point of the area at which R acts. Then $\bar{x}, \bar{y}, R$, may be determined from the three equations

$$R = g\rho\Sigma\,(hK), \quad \bar{x}.R = g\rho\Sigma\,(hxK), \quad \bar{y}.R = g\rho\Sigma\,(hyK),$$

the formulæ for $\bar{x}, \bar{y}$, being

$$\bar{x} = \frac{\Sigma\,(hxK)}{\Sigma\,(hK)}, \quad \bar{y} = \frac{\Sigma\,(hyK)}{\Sigma\,(hK)},$$

the limits of the summations being dependent upon the form of the immersed area. The point of which $\bar{x}$ and $\bar{y}$ are the coordinates is called the *Centre of Pressure.*

If the axes of coordinates be at right angles to each other, $g\rho\Sigma\,(hxK)$ and $g\rho\Sigma\,(hyK)$ denote the moments of the pressure of the fluid about the axis of y and x respectively.

If the axes of coordinates be at right angles to each other, and that of y be in the surface of the fluid, then, θ being the inclination of the immersed area to the vertical,

$$h = x \cos \theta, \quad K = dxdy,$$

and therefore

$$\bar{x} = \frac{\Sigma (x^2 K)}{\Sigma (xK)}, \quad \bar{y} = \frac{\Sigma (xyK)}{\Sigma (xK)};$$

these formulæ shew that, if the immersed area be a plane lamina of uniform thickness, moveable about the axis of y, the *Centre of Pressure* will coincide with the *Centre of Percussion*, a proposition demonstrated by Cotes, (*Hydrostatical and Pneumatical Lectures*, pp. 41, 42, 3rd. edit.).

If we adopt the notation of the Integral Calculus, we have, putting $K = dx\,dy$,

$$\bar{x} = \frac{\iint hx\,dx\,dy}{\iint h\,dx\,dy}, \quad \bar{y} = \frac{\iint hy\,dx\,dy}{\iint h\,dx\,dy};$$

the limits of the integration depending upon the boundary of the area. If the boundary of the area be discontinuous, we must divide the area into a number of portions with continuous boundaries; we must then take the sum of the values of each of the integrals

$$\iint x\,dx\,dy, \quad \iint x^2\,dx\,dy, \quad \iint xy\,dx\,dy,$$

for these several portions, to express its value for the whole of the immersed area. It is frequently more convenient to adopt polar formulæ.

If the axes of coordinates be rectangular, that of y being parallel to the surface of the fluid, and the area immersed be symmetrical with regard to the axis of x, it is plain that the *Centre of Pressure* must lie in the axis of x, and it is therefore necessary to evaluate only the two expressions $\Sigma\,(hK)$ and $\Sigma\,(hxK)$.

The idea of a *Centre of Pressure* on a plane surface subjected to the pressure of a fluid, was first developed by Stevin in his *Beghinselen der Waaghconst*, published in 1585. The existing state of analysis rendered it impossible to make any but the most elementary applications of his conception. He has actually calculated however the position the centre of pressure of a

parallelogram immersed to any depth in a fluid, with one edge parallel to the surface, the plane of the parallelogram being inclined at any angle whatever to the horizon, and has laid down on correct mechanical principles a general sketch of the method of extending such a computation to any plane rectilinear figures whatever. These investigations may be seen in the French translation of Stevin's works by Albert Girard, pp. 495, 496, 497. On the discovery of the principles of the infinitesimal analysis, the determination of the centres of pressure of planes of various forms was effected without difficulty. Among the earliest writers who have treated on the problem of the centre of pressure may be mentioned Herman, *Phoronomia*, p. 141, anno 1716, and Cotes, *Hydrostatical and Pneumatical Lectures*, pp. 40, 3rd. edit.

1. To find the centre of pressure of a parallelogram immersed in a fluid, one edge of the parallelogram being in the surface.

Let $ABCD$ (fig. 4) be the parallelogram, OE being a straight line joining the middle points O, E, of the two horizontal sides. Let OE, OB, produced indefinitely to x, y, be taken as the axes of coordinates. Let PP', pp', be two lines parallel to AB, cutting OE in two points M, m, very near to each other. Then, α being the inclination of Ox to Oy,

$$K = 2y\,dx \,.\sin\alpha,$$

$$h = x\sin\alpha;$$

and therefore, if $OE = l$,

$$\int_0^l 2y\,dx \,.\sin\alpha \,.\, x\sin\alpha \,.\, x = \bar{x}\int_0^l 2y\,dx\sin\alpha \,.\, x\sin\alpha,$$

or, y being invariable,

$$\int_0^l x^2\,dx = \bar{x}\int_0^l x\,dx,$$

$$\tfrac{1}{3}l^3 = \tfrac{1}{2}l^2 \,.\, \bar{x},$$

$$\bar{x} = \tfrac{2}{3}l,$$

which shews that, if G be the centre of pressure,

$$OG = 2GE.$$

"*Si le fond d'une eau n'est à niveau, estant parallelogramme, duquel le plus haut costé soit à fleur d'eau, et de son milieu, au*

milieu de son costé opposite, est menée une ligne; le centre de gravité (du pressement de l'eau congregé contre le fond) divise ceste ligne de telle sorte, que la partie haute à la basse est en raison double."

Stevin: *Œuvres Mathématiques par Albert Girard,* p. 495.

2. To find the centre of pressure of a portion of a parabolic area bounded by the axis of the parabola, the curve, and an ordinate at right angles to the axis, supposing the ordinate to lie in the surface of the fluid.

Let a denote the portion of the axis between the foot of the ordinate and the vertex of the parabola, and b the length of the ordinate. Let the axis of x coincide with the axis of the parabola, and the axis of y with the tangent at its vertex, and let θ be the inclination of the axis of the parabola to the vertical.

Then, supposing the parabola to be divided into small strips, contained between consecutive ordinates, we have

$$K = y\,dx, \qquad h = (a - x)\cos\theta;$$

and therefore $$\bar{x}\int_0^a y\,dx\,.\,(a-x) = \int_0^a xy\,dx\,.\,(a-x);$$

$$\bar{x}\int_0^a (a-x)\,x^{\frac{1}{2}}dx = \int_0^a (a-x)\,x^{\frac{3}{2}}dx,$$

$$\bar{x}\,.\left(\tfrac{2}{3}a\,.\,a^{\frac{3}{2}} - \tfrac{2}{5}a^{\frac{5}{2}}\right) = \tfrac{2}{5}a\,.\,a^{\frac{5}{2}} - \tfrac{2}{7}a^{\frac{7}{2}},$$

$$\tfrac{4}{15}\bar{x} = \tfrac{4}{35}a, \quad \bar{x} = \tfrac{3}{7}a.$$

Again, dividing the parabolic area into elementary rectangles $dx\,dy$, we have

$$\bar{y}\int_0^y\int_0^a (a-x)\,dx\,dy = \int_0^y\int_0^a (a-x)\,y\,dx\,dy,$$

$$\bar{y}\int_0^a (a-x)\,y\,dx = \tfrac{1}{2}\int_0^a (a-x)\,y^2dx;$$

or, $4m$ being the latus rectum of the parabola,

$$\bar{y}\int_0^a (a-x)\,x^{\frac{1}{2}}dx = m^{\frac{1}{2}}\int_0^a (a-x)\,x\,dx,$$

$$\bar{y}\left(\tfrac{2}{3}a\,.\,a^{\frac{3}{2}} - \tfrac{2}{5}a^{\frac{5}{2}}\right) = m^{\frac{1}{2}}\left(\tfrac{1}{2}a\,.\,a^2 - \tfrac{1}{3}a^3\right),$$

$$\bar{y}\,.\left(\tfrac{2}{3} - \tfrac{2}{5}\right) = (ma)^{\frac{1}{2}}.\left(\tfrac{1}{2} - \tfrac{1}{3}\right),$$

$$\bar{y} = \tfrac{15}{24}(ma)^{\frac{1}{2}} = \tfrac{5}{16}b.$$

3. To find the centre of pressure of a triangle immersed vertically to any depth beneath the surface of a fluid, the base of the triangle being horizontal and its vertex being the point of the triangle which is nearest to the surface.

Let c denote the depth of the vertex of the triangle below the surface, h the altitude of the vertex above the base. Let x be the depth, below the vertex, of any line y drawn parallel to the base of the triangle and intercepted by the other two sides. Then, $c + x$ being the depth of the elemental strip $y\,dx$ below the surface, we have, supposing $\bar{x}$ to be the depth of the centre of pressure below the vertex,

$$\bar{x} \,.\int_0^h (x + c)\, y\, dx = \int_0^h (x+c)\, xy\, dx;$$

or, since y varies directly as x,

$$\bar{x} \,.\int_0^h (x + c)\, x\, dx = \int_0^h (x + c)\, x^2\, dx,$$

$$\bar{x} \,.(\tfrac{1}{3} h^3 + \tfrac{1}{2} ch^2) = \tfrac{1}{4} h^4 + \tfrac{1}{3} ch^3,$$

$$\bar{x} = h \,.\, \frac{3h + 4c}{4h + 6c}.$$

Since the line y is always bisected by a line passing through the vertex of the triangle and the middle point of its base, it is obvious that the centre of pressure must lie in this bisecting line. Its distance from the vertex, if l denote the length and θ the inclination of the bisecting line to the horizon, will be equal to

$$\bar{x} \operatorname{cosec} \theta = \operatorname{cosec} \theta \,.\, h \,.\, \frac{3h + 4c}{4h + 6c} = l\, \frac{3h + 4c}{4h + 6c}.$$

4. To find the centre of pressure of a square AB (fig. 5) immersed to a given depth below the surface of a fluid, one of its diagonals OC being vertical.

Let the diagonal CO produced meet the surface KDL in the point D, and let $OD = h$; let OA, OB, produced indefinitely be the axes of x and y. Let x, y, be the coordinates of a point P of the square, and $x + dx$, $y + dy$, of a point p indefinitely

near to it; then the depth of the elementary square, of which Pp is a diagonal, below the surface of the fluid, being

$$h + \frac{x+y}{\sqrt{2}},$$

and its area being $dx\,dy$, we have, a denoting a side of the square AB,

$$\bar{x}\int_0^a\int_0^a\left(h + \frac{x+y}{\sqrt{2}}\right)dx\,dy = \int_0^a\int_0^a\left(h + \frac{x+y}{\sqrt{2}}\right)x\,dx\,dy.$$

The coefficient of $\bar{x}$ in this equation is equal to

$$\int_0^a\left(ha + \frac{ax}{\sqrt{2}} + \frac{a^2}{2\sqrt{2}}\right)dx = ha^2 + \frac{a^3}{2\sqrt{2}} + \frac{a^3}{2\sqrt{2}} = \frac{a^2}{\sqrt{2}}(h\sqrt{2} + a),$$

and the right-hand side of the equation is equal to

$$\int_0^a\left(\tfrac{1}{2}ha^2 + \frac{a^2y}{2\sqrt{2}} + \frac{a^3}{3\sqrt{2}}\right)dy = \tfrac{1}{2}ha^3 + \frac{a^4}{4\sqrt{2}} + \frac{a^4}{3\sqrt{2}} = \frac{a^3}{12\sqrt{2}}(6h\sqrt{2} + 7a).$$

Hence $$\bar{x} = \frac{a}{12}\cdot\frac{6h\sqrt{2} + 7a}{h\sqrt{2} + a}.$$

5. An area bounded by the curve

$$r = a(1 + \cos\theta),$$

is immersed vertically in a fluid, the prime radius vector being coincident with the surface; to find the depth of the centre of pressure.

The required depth being denoted by $\bar{x}$, we must determine $\bar{x}$ from the equation

$$\bar{x}\iint r\,d\theta\,dr\,.\,r\sin\theta = \iint r\,d\theta\,dr\,.\,r^2\sin^2\theta,$$

$r\sin\theta$ being the depth of the element $r\,d\theta\,dr$ below the surface. Indicating the limits,

$$\bar{x}\int_0^r\int_0^\pi r^2\sin\theta\,d\theta\,dr = \int_0^r\int_0^\pi r^3\sin^2\theta\,d\theta\,dr,$$

$$\tfrac{1}{3}\bar{x}\int_0^\pi r^3\sin\theta\,d\theta = \tfrac{1}{4}\int_0^\pi r^4\sin^2\theta\,d\theta,$$

$$\tfrac{1}{3}\bar{x}\int_0^\pi(1 + \cos\theta)^3\sin\theta\,d\theta = \tfrac{1}{4}a\int_0^\pi(1 + \cos\theta)^4\sin^2\theta\,d\theta.$$

Now $$\int_0^\pi(1 + \cos\theta)^3\sin\theta\,d\theta = {}_0^\pi\{-\tfrac{1}{4}(1 + \cos\theta)^4\} = 4.$$

Also

$$(1+\cos\theta)^4\sin^2\theta d\theta=(1+4\cos\theta+6\cos^2\theta+4\cos^3\theta+\cos^4\theta)\sin^2\theta d\theta$$
$$=\tfrac{1}{2}(1-\cos 2\theta)\,d\theta+4\sin^2\theta\,d\sin\theta+\tfrac{3}{4}(1-\cos 4\theta)\,d\theta$$
$$+\,4(\sin^2\theta-\sin^4\theta)\,d\sin\theta+\sin^2\theta\cos^4\theta d\theta,$$

and therefore

$$\int_0^\pi(1+\cos\theta)^4\sin^2\theta d\theta=\tfrac{1}{2}\pi+\tfrac{3}{4}\pi+\int_0^\pi\sin^2\theta\cos^4\theta d\theta$$
$$=\tfrac{5}{4}\pi+\int_0^\pi\sin^2\theta\cos^4\theta d\theta;$$

but, integrating by parts,

$$\int_0^\pi\sin^2\theta\cos^4\theta\,d\theta=-\tfrac{1}{5}\,\Big|_0^\pi\Big(\sin\theta\cos^5\theta\Big)+\tfrac{1}{5}\int_0^\pi\cos^6\theta d\theta$$
$$=\tfrac{1}{5}\int_0^\pi\cos^4\theta\,(1-\sin^2\theta)\,d\theta$$
$$=\tfrac{1}{6}\int_0^\pi\cos^4\theta d\theta,$$

$$\int_0^\pi\cos^4\theta d\theta=\Big|_0^\pi\Big(\cos^3\theta\sin\theta\Big)+3\int_0^\pi\cos^2\theta\sin^2\theta d\theta$$
$$=\tfrac{3}{4}\int_0^\pi\cos^2\theta d\theta=\tfrac{3}{8}\int_0^\pi(1+\cos 2\theta)\,d\theta=\tfrac{3}{8}\pi;$$

and therefore

$$\int_0^\pi(1+\cos\theta)^4\sin^2\theta d\theta=\tfrac{5}{4}\pi+\frac{\pi}{16}=\frac{21\pi}{16}.$$

Hence
$$\tfrac{1}{3}\bar{x}\,.\,4=\tfrac{1}{4}a\,.\,\frac{21\pi}{16}$$
$$\bar{x}=\frac{63\pi a}{256}$$

6. An elliptic area PCD, bounded by two conjugate semi-diameters CP, CD, and the intercepted arc, is immersed vertically in a fluid, CD coinciding with the surface; to find the centre of pressure.

Let CP and CD be taken as the axes of x and y respectively. Let α denote the angle between CP and CD; let $CP=a$, $CD=b$.

Then the area of an elementary parallelogram, the sides of

which are dx, dy, parallel to the axes of coordinates, will be $dxdy \sin \alpha$, and its depth below the surface will be $x \sin \alpha$. Hence $\bar{x}$ and $\bar{y}$ are to be found from the equations

$$\bar{x} \iint dxdy \sin \alpha \,.\, x \sin \alpha = \iint dxdy \sin \alpha \,.\, x \sin \alpha \,.\, x,$$

or, reducing and indicating the limits,

$$\bar{x} \int_0^y \int_0^a x\,dx\,dy = \int_0^y \int_0^a x^2\,dx\,dy \,\ldots\ldots\ldots\ldots (1),$$

and $\quad \bar{y} \iint dxdy \sin \alpha \,.\, x \sin \alpha = \iint dxdy \sin \alpha \,.\, x \sin \alpha \,.\, y,$

or

$$\bar{y} \int_0^y \int_0^a x\,dx\,dy = \int_0^y \int_0^a xy\,dx\,dy \ldots\ldots\ldots\ldots (2).$$

Now

$$\int_0^y \int_0^a x\,dx\,dy = \int_0^a xy\,dx = \frac{b}{a} \int_0^a x\,(a^2 - x^2)^{\frac{1}{2}} dx$$

$$= \frac{b}{a} \,.\, {}_0^a\left\{- \tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right\} = \tfrac{1}{3} a^2 b.$$

Also

$$\int_0^y \int_0^a x^2\,dx\,dy = \frac{b}{a} \int_0^a x^2\,(a^2 - x^2)^{\frac{1}{2}}\,dx\,;$$

but $\displaystyle\int_0^a x^2\,(a^2 - x^2)^{\frac{1}{2}}\,dx = {}_0^a\left\{- \tfrac{1}{3} x\,(a^2 - x^2)^{\frac{3}{2}}\right\} + \tfrac{1}{3} \int_0^a (a^2 - x^2)^{\frac{3}{2}}\,dx\,;$

$$= \tfrac{1}{3} a^2 \int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx - \tfrac{1}{3} \int_0^a x^2\,(a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{4} a^2 \int_0^a (a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{4} a^2 \,.\, \tfrac{1}{4} \pi a^2 = \tfrac{1}{16} \pi a^4\,;$$

and therefore

$$\int_0^y \int_0^a x^2\,dx\,dy = \tfrac{1}{16} \pi a^3 b.$$

Again

$$\int_0^y \int_0^a xy\,dx\,dy = \tfrac{1}{2} \int_0^a \frac{b^2}{a^2}\,(a^2 - x^2)\,x\,dx$$

$$= \tfrac{1}{2} \frac{b^2}{a^2} \left(\tfrac{1}{2} a^4 - \tfrac{1}{4} a^4\right)$$

$$= \tfrac{1}{8} a^2 b^2.$$

Hence, from (1), $\quad \bar{x} \,.\, \tfrac{1}{3} a^2 b = \tfrac{1}{16} \pi a^3 b,$

and therefore $\quad \bar{x} = \tfrac{3}{16} \pi a,$

and, from (2), $\quad \bar{y} \,.\, \tfrac{1}{3} a^2 b = \tfrac{1}{8} a^2 b^2,$

and therefore $\quad \bar{y} = \tfrac{3}{8} b.$

7. A rectangular flood-gate is moveable about its lowest edge as a horizontal axis; having given the depth of the water on each side, to determine the moment of the couple which will keep it at rest.

Let c, c', represent the altitudes of the surfaces of the water on the two sides of the gate above its lowest edge: let b denote the breadth of the gate. Then, x being the depth of any point of the gate below the surface on one side, the moment exerted by the water on this side to produce rotation about the axis is equal to

$$\int_0^c g\rho x \,.\, b dx \,.\, (c - x)$$

$$= y\rho b \,.\, \left(\tfrac{1}{2}c^3 - \tfrac{1}{3}c^3\right) = \tfrac{1}{6} g\rho b c^3.$$

The opposite moment due to the fluid on the other side will accordingly, putting c' instead of c, be equal to

$$\tfrac{1}{6} g\rho b c'^3.$$

Hence the moment of the required couple will be equal to

$$\tfrac{1}{6} g\rho c \left(c'^3 \sim c^3\right).$$

8. To find the centre of pressure of a rectangular plank immersed vertically to any depth within a fluid, the two ends of the plank being horizontal.

If a be the depth of the upper and b of the lower end of the plank, the depth of the centre of pressure will be equal to

$$\frac{2}{3} \cdot \frac{b^3 - a^3}{b^2 - a^2}.$$

Stevin: *Œuvres Mathématiques par Albert Girard*, p. 496.

9. A circular area is just immersed in a fluid; to find the depth of the centre of pressure below the centre of the area.

If r denote the radius of the circle, the required depth will be equal to $\frac{1}{4}r$.

10. To find the position of the centre of pressure of an equilateral triangle having one angle in the surface of the fluid and one side vertical.

Let l represent the length of a side of the triangle, and h the distance of one of its angles from the opposite side; then, the

angle of the triangle opposite to the vertical side being taken as origin of coordinates, the axis of x being taken horizontal so as to bisect the vertical side, and the axis of y being taken vertically downwards,

$$\bar{x} = \tfrac{2}{3}h, \quad \bar{y} = \tfrac{1}{12}\,l.$$

11. To find the centre of pressure of a segment of a parabola cut off by a chord passing through the vertex, the tangent at which lies in the surface of the fluid, and inclined at an angle of 45° to the axis.

The axis of the parabola and the tangent at its vertex being taken as the axes of x and y respectively, and l representing the latus rectum,

$$\bar{x} = \tfrac{15}{28}l, \quad \bar{y} = \tfrac{5}{8}l.$$

12. A parabolic area (fig. 6), cut off by a single ordinate, the abscissa of which is equal to the latus rectum, is immersed vertically in fluid, its axis being inclined at an angle of 30° to the horizon, and its vertex being at a distance equal to the latus rectum below the surface: to determine the position of the centre of pressure.

Let the axis of the parabola be taken as the axis of x, and the tangent at its vertex as the axis of y, and let l denote the latus rectum. Then

$$\bar{x} = l.\frac{\frac{19}{35} - \frac{1}{4\sqrt{3}}}{\frac{13}{15} - \frac{\sqrt{3}}{8}}, \quad \bar{y} = l.\frac{\frac{1}{3} - \frac{1}{5\sqrt{3}}}{\frac{13}{15} - \frac{\sqrt{3}}{8}}.$$

13. An area in the form of a sector of an equilateral hyperbola, bounded by the axis, a line through the centre, and the curve, is placed in a fluid with the axis in its surface: to find the depth of the centre of pressure.

If a be the semi-axis of the hyperbola, and α the angle between the two radii vectores of the sector, the required depth will be equal to

$$\tfrac{3}{16}a\,.\,\frac{\tan 2\alpha + \log(1 - \tan\alpha) - \log(1 + \tan\alpha)}{\cos\alpha\,.\,(\sec 2\alpha)^{\frac{3}{2}} - 1}.$$

14. Two equal parabolas, with the same vertex, having their axes at right angles to each other, are immersed vertically in a fluid, the axis of one of them coinciding with the surface: to determine the position of the centre of pressure of the area which is common to both.

Let l represent the latus rectum of each parabola, then, the axis of x extending vertically downwards in coincidence with the axis of one parabola, and the axis of y coinciding in the surface of the fluid with the axis of the other parabola,

$$\bar{x} = \tfrac{4}{7}l, \quad \bar{y} = \tfrac{5}{9}l.$$

15. A flood-gate ADO (fig. 7) moves upon a vertical axis AO, the area ADO, on one side of the axis, being the quadrant of a circle, and, on the other side, a rectangle $ABCO$ of the same altitude: to determine the width AB of the parallelogram so that the gate may just open by the pressure of the water when it has risen to the top.

If a represent the radius of the circle, and x the width of the rectangle,

$$x = \frac{a}{\sqrt{2}}.$$

16. A hollow cube, just filled with heavy fluid, is held with one diagonal vertical: to find the centre of pressure of one of the lower faces.

If a denote the length of each edge of the cube, and if the lowest point of the cube be taken as the origin of coordinates, the axes of coordinates being two sides of a lower face, the centre of pressure of this face is given by the equations

$$\bar{x} = \tfrac{11}{24}a = \bar{y}.$$

17. The axis of a cylindrical vessel, containing a known quantity of fluid, is inclined at a given angle to the horizon: to determine the centre of pressure of its base.

If r represent the radius of the cylinder, c^3 the known quantity of fluid, and α the inclination of the axis of the cylinder

to the horizon, the distance of the centre of pressure from the geometrical centre of the base is equal to

$$\frac{\pi r^4}{4c^3 \tan \alpha}.$$

Section III.

Heterogeneous Incompressible Fluids.

If the fluid be heterogeneous, let p denote the unit of pressure at any elemental area K, then, the notation of the preceding section being retained, we shall have, instead of the formulæ there given,

$$\bar{x} = \frac{\Sigma\,(pKx)}{\Sigma\,(pK)}, \quad \bar{y} = \frac{\Sigma\,(pKy)}{\Sigma\,(pK)}.$$

If, with the increase of the depth, the density of the fluid and therefore the unit of pressure vary discontinuously, we must subdivide the fluid into a series of horizontal strata, such that the law of the variation of the density and unit of pressure shall be continuous throughout each of them. We must then take the sum of the values of each of the expressions

$$\Sigma\,(pK), \quad \Sigma\,(pKx), \quad \Sigma\,(pKy),$$

for the several strata to which the plane surface is exposed, as constituting its value in the preceding formulæ.

1. To find the centre of pressure of a parallelogram OC (fig. 8) immersed vertically in a fluid with one angle O at the surface, the density of the fluid being supposed to vary as the depth.

Let α, β, be the inclinations of the sides OA, OB, to the vertical; let $OA = a$, $OB = b$. Let Pp be any elemental parallelogram in the area OC, its sides dx, dy, being parallel to the axes of x, y, respectively, that is, to the sides OA, OB, produced indefinitely.

Then, h being the depth of P below the surface,

$$h = x \cos \alpha + y \cos \beta,$$

and, k denoting a constant quantity,

$$p = \int_0^h g\rho dh = kg\int_0^h hdh = \tfrac{1}{2}kgh^2;$$

whence $$p = \tfrac{1}{2}kg\,(x\cos\alpha + y\cos\beta)^2.$$

Thus, $dx\,dy\sin(\alpha+\beta)$ being the area of Pp,

$$\bar{x}\int_0^b\!\!\int_0^a (x\cos\alpha + y\cos\beta)^2\,dx\,dy = \int_0^b\!\!\int_0^a (x\cos\alpha + y\cos\beta)^2 x\,dx\,dy.$$

Now $\int_0^b (x\cos\alpha + y\cos\beta)^2\,dy = bx^2\cos^2\alpha + b^2x\cos\alpha\cos\beta + \tfrac{1}{3}b^3\cos^2\beta,$

$$\int_0^b\!\!\int_0^a (x\cos\alpha + y\cos\beta)^2\,dx\,dy = b(\tfrac{1}{3}a^3\cos^2\alpha + \tfrac{1}{2}a^2b\cos\alpha\cos\beta + \tfrac{1}{3}ab^2\cos^2\beta),$$

$$\int_0^b\!\!\int_0^a (x\cos\alpha + y\cos\beta)^2\,x\,dx\,dy$$
$$= b\,(\tfrac{1}{4}a^4\cos^2\alpha + \tfrac{1}{3}a^3b\cos\alpha\cos\beta + \tfrac{1}{6}a^2b^2\cos^2\beta):$$

hence

$$\bar{x}\,.\,(\tfrac{1}{3}a^2\cos^2\alpha + \tfrac{1}{2}ab\cos\alpha\cos\beta + \tfrac{1}{3}b^2\cos^2\beta)$$
$$= (\tfrac{1}{4}a^2\cos^2\alpha + \tfrac{1}{3}ab\cos\alpha\cos\beta + \tfrac{1}{6}b^2\cos^2\beta)\,a,$$

$$\bar{x}\,.\,(4a^2\cos^2\alpha + 6ab\cos\alpha\cos\beta + 4b^2\cos^2\beta)$$
$$= a\,.\,(3a^2\cos^2\alpha + 4ab\cos\alpha\cos\beta + 2b^2\cos^2\beta).$$

Similarly

$$\bar{y}\,.\,(4b^2\cos^2\beta + 6ba\cos\beta\cos\alpha + 4a^2\cos^2\alpha)$$
$$= b\,(3b^2\cos^2\beta + 4ba\cos\beta\cos\alpha + 2a^2\cos^2\alpha).$$

2. To find the centre of pressure of a rectangular board immersed vertically in a vessel containing three fluids of different densities, the length of the board being just equal to the whole depth of the fluid and being divided by the three fluid strata into three equal parts.

Let a be the length of each portion of the board, and let ρ, ρ', ρ'', be the densities of the three fluids, reckoning from the highest. Then, at a depth x below the edge of the higher extremity of the board,

$p = g\rho x$, in the highest fluid;

$p = g\rho a + g\rho'(x - a)$, in the middle fluid;

$p = g\rho a + g\rho' a + g\rho''(x - 2a)$, in the lowest fluid.

Hence, m denoting the breadth of the board, we have, putting mdx for K,

$$\Sigma_0^a(pK) = g\rho m \int_0^a x\,dx = \tfrac{1}{2}g\rho ma^2:$$

$$\Sigma_a^{2a}(pK) = gm \int_a^{2a} (\rho a - \rho' a + \rho' x)\,dx$$

$$= gma^2(\rho + \tfrac{1}{2}\rho'):$$

$$\Sigma_{2a}^{3a}(pK) = gm \int_{2a}^{3a} \{\rho a + \rho' a - 2\rho'' a + \rho'' x\}\,dx$$

$$= gma^2(\rho + \rho' + \tfrac{1}{2}\rho''):$$

and therefore

$$\Sigma_0^{3a}(pK) = \tfrac{1}{2}gma^2(5\rho + 3\rho' + \rho'').$$

Again $\Sigma_0^a(pKx) = g\rho m \int_0^a x^2dx = \tfrac{1}{3}g\rho ma^3:$

$$\Sigma_a^{2a}(pKx) = gm \int_a^{2a} (\rho a - \rho' a + \rho' x)\,x\,dx$$

$$= gma^3(\tfrac{3}{2}\rho + \tfrac{5}{6}\rho'):$$

$$\Sigma_{2a}^{3a}(pKx) = gm \int_{2a}^{3a} (\rho a + \rho' a - 2\rho'' a + \rho'' x)\,x\,dx$$

$$= gma^3(\tfrac{5}{2}\rho + \tfrac{5}{2}\rho' + \tfrac{4}{3}\rho''):$$

and therefore

$$\Sigma_0^{3a}(pKx) = \tfrac{1}{3}gma^3(13\rho + 10\rho' + 4\rho'').$$

Hence
$$\bar{x} = \tfrac{2}{3}a \,.\, \frac{13\rho + 10\rho' + 4\rho''}{5\rho + 3\rho' + \rho''}.$$

3. A semicircular area is just immersed in a fluid, its diameter being horizontal; the density of the fluid varies as the depth: to determine the centre of pressure.

The centre of pressure will be in the vertical radius of the semicircle, and, a denoting the radius, at a distance from the centre equal to

$$\frac{32\,a}{15\,\pi}.$$

4. A segment of a parabola, cut off by a double ordinate to the axis, is immersed in a fluid the density of which varies as

the depth; the tangent at the vertex of the segment lies in the surface of the fluid: to find the depth of the centre of pressure.

If h represent the length of the axis of the parabolic segment, the required depth will be equal to $\frac{7}{9}h$.

5. A segment of a parabola, cut off by a double ordinate to the axis, is immersed in a fluid, the density of which varies as the depth; the base of the segment lies in the surface of the fluid: to find the centre of pressure.

If h denote the length of the axis of the segment, and $\bar{x}$ the altitude of the centre of pressure above the vertex,

$$\bar{x} = \tfrac{1}{3}h.$$

CHAPTER III.

RESULTANT PRESSURES OF FLUIDS ON CURVED SURFACES OF IMMERSED SOLIDS.

LET K be any indefinitely small element of the surface of a solid at a depth h below the surface of a homogeneous incompressible fluid, and let ρ be the density of the fluid. Let A, B, C, be the areas of the projections of K upon the coordinate planes yz, zx, xy, respectively, the axes of coordinates being supposed to be rectangular. Let X, Y, Z, be the components of the resultant pressure, and $\bar{x}$, $\bar{y}$, $\bar{z}$, the coordinates of any point in its direction.

Then, the limits of the summation being defined by the boundary of the surface,

$$X = g\rho\Sigma(hA), \quad Y = g\rho\Sigma(hB), \quad Z = g\rho\Sigma(hC),$$

$$Z\bar{y} - Y\bar{z} = g\rho\{\Sigma(hCy) - \Sigma(hBz)\} = L,$$

$$X\bar{z} - Z\bar{x} = g\rho\{\Sigma(hAz) - \Sigma(hCx)\} = M,$$

$$Y\bar{x} - X\bar{y} = g\rho\{\Sigma(hBx) - \Sigma(hAy)\} = N:$$

from the first three of these equations we can determine the magnitude of the resultant; and, eliminating X, Y, Z, we have for the equations to its direction, any two of the three equations

$$\bar{y}\Sigma(hC) - \bar{z}\Sigma(hB) = \Sigma(hCy) - \Sigma(hBz),$$

$$\bar{z}\Sigma(hA) - \bar{x}\Sigma(hC) = \Sigma(hAz) - \Sigma(hCx),$$

$$\bar{x}\Sigma(hB) - \bar{y}\Sigma(hA) = \Sigma(hBx) - \Sigma(hAy).$$

In these formulæ, whenever X acts in the negative instead of the positive direction, we must replace A by $-A$: analogous remarks are applicable to the actions of Y and Z.

Multiplying L, M, N, by X, Y, Z, respectively, we see that

$$LX + MY + NZ = 0,$$

which is the condition necessary that the three equations defining the direction of the resultant force may coexist; in other words, this relation is the condition that the pressures on the elements of the surface may be all reducible to a single resultant.

If the fluid be a heterogeneous fluid, then, p denoting the unit of pressure at K, we must replace the six preceding equations for the determination of the resultant pressure by the six following:

$$X = \Sigma(pA), \quad Y = \Sigma(pB), \quad Z = \Sigma(pC);$$

$$\bar{y}\Sigma(pC) - \bar{z}\Sigma(pB) = \Sigma(pCy) - \Sigma(pBz),$$

$$\bar{z}\Sigma(pA) - \bar{x}\Sigma(pC) = \Sigma(pAz) - \Sigma(pCx),$$

$$\bar{x}\Sigma(pB) - \bar{y}\Sigma(pA) = \Sigma(pBx) - \Sigma(pAy):$$

the value of p in these equations, if the fluid be incompressible, will be some function of h, and, ρ being some given function of h, may be determined from the equation

$$dp = g\rho dh:$$

if the fluid be elastic, then, g denoting the force of gravity, k a constant quantity dependent upon the elasticity of the fluid, and C an arbitrary constant which may be determined when simultaneous values of h and p are given,

$$p = C \,.\, \epsilon^{\frac{gh}{k}}.$$

If g be very small compared with $\frac{k}{h}$, or the weight of the fluid be inconsiderable in comparison with its elastic force, then $p = C$ very nearly, or the unit of pressure is the same at every point.

Let S represent the magnitude of any area immersed in a fluid; through the periphery of S conceive an indefinite number of vertical lines to be drawn so as to trace out on the horizontal plane coinciding with the surface of the fluid a curvilinear figure including an area of magnitude S'. Also let V denote

the volume included between these vertical lines and the areas S, S'. Then, in accordance with the equations

$$Z = \Sigma(pC), \quad p = g\int\rho\, dh,$$

we see that the vertical pressure upon S is equal to the weight of the volume V of the fluid.

1. A vessel, bounded by two vertical planes at right angles to each other, and a paraboloid of revolution, the axis of which coincides with the intersection of the planes, is filled with fluid as far as the focus: to determine the magnitude and direction of the resultant pressure.

Let the vertex of the paraboloid be taken as the origin of coordinates, and its axis as the axis of z, the plane of xz being supposed to bisect the right angle contained between the two vertical planes. It is evident that, the surface being symmetrical with regard to the plane of xz, the resultant pressure must lie within the plane xz. Hence it will be necessary to take only the equations

$$X = g\rho\Sigma(hA), \quad Z = -g\rho\Sigma(hC),$$

$$\bar{z}\Sigma(hA) + \bar{x}\Sigma(hC) = \Sigma(hAz) + \Sigma(hCx).$$

Let $4m$ be the latus rectum of the paraboloid; then $h = m - z$, and therefore, the integrations being performed from $y = -(2mz)^{\frac{1}{2}}$ to $y = (2mz)^{\frac{1}{2}}$, and from $z = 0$ to $z = m$,

$$\Sigma(hA) = \iint(m - z)\, dy\, dz$$

$$= 2(2m)^{\frac{1}{2}}\int_0^m (mz^{\frac{1}{2}} - z^{\frac{3}{2}})\, dz = \tfrac{8}{15}m^3\sqrt{2},$$

$$\Sigma(hAz) = \iint(m - z)\, z.dy\, dz = 2(2m)^{\frac{1}{2}}\int_0^m (mz^{\frac{3}{2}} - z^{\frac{5}{2}})\, dz = \tfrac{8}{35}m^4\sqrt{2}.$$

Again, r being the distance of the small area C from the origin of coordinates, and θ the inclination of r to the axis of x,

$$\Sigma(hC) = \iint(m - z)\, r\, d\theta\, dr$$

$$= \iint\left(m - \frac{r^2}{4m}\right) r\, d\theta\, dr,$$

the integration being performed from $r = 0$ to $r = 2m$, and from $\theta = -\frac{1}{4}\pi$ to $\theta = \frac{1}{4}\pi$: hence

$$\Sigma(hC) = m^3\int d\theta = \tfrac{1}{2}\pi m^3:$$

also, between the same limits,

$$\Sigma\,(hCx) = \int\!\!\int (m - z)\, xr d\theta\, dr$$

$$= \int\!\!\int \left(m - \frac{r^2}{4m}\right) r \cos\theta \,.\, r d\theta\, dr$$

$$= \tfrac{16}{15}\, m^4 \int \cos\theta\, d\theta$$

$$= \frac{16\sqrt{2}}{15}\, m^4.$$

Hence, R denoting the resultant pressure,

$$R = (X^2 + Z^2)^{\frac{1}{2}} = g\rho m^3 \left(\frac{\pi^2}{4} + \frac{128}{225}\right)^{\frac{1}{2}};$$

and the equations to its direction will be

$$\bar{y} = 0,$$

and $$\frac{8\sqrt{2}}{15}\,\bar{z} + \tfrac{1}{2}\pi\bar{x} = \frac{136\sqrt{2}}{105}\, m.$$

2. A hemispheroidal bowl is filled with fluid; to find the magnitude and direction of the pressure upon a quarter of the surface bounded by two planes passing through the axis, which is vertical.

Let the axis of the bowl be taken as the axis of z, and the intersections of the two vertical planes with the plane bounded by the rim of the bowl, the axes of x and y. Then a and c being the semi-axes of the generating ellipse, the equation to the surface will be

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

Hence, integrating from $z = 0$ to $z = \frac{c}{a}(a^2 - y^2)^{\frac{1}{2}}$, and from $y = 0$ to $y = a$, we have

$$\Sigma\,(hA) = \int\!\!\int z\, dy\, dz = \tfrac{1}{2}\int z^2 dy = \tfrac{1}{2}\,\frac{c^2}{a^2}\int (a^2 - y^2)\, dy = \tfrac{1}{3} ac^2.$$

Also $\Sigma(hAz) = \int\!\!\int z^2\, dy\, dz = \frac{1}{3}\int z^3 dy = \frac{1}{3}\,\frac{c^3}{a^3}\int_0^a (a^2 - y^2)^{\frac{3}{2}} dy$:

but $\int_0^a (a^2 - y^2)^{\frac{3}{2}} dy = {}_0^a\left\{(a^2 - y^2)^{\frac{3}{2}} y\right\} + 3\int_0^a (a^2 - y^2)^{\frac{1}{2}} y^2 dy$

$$= -3\int_0^a (a^2 - y^2)^{\frac{3}{2}} dy + 3a^2 \int_0^a (a^2 - y^2)^{\frac{1}{2}} dy$$

$$= \tfrac{3}{4} a^2 \int_0^a (a^2 - y^2)^{\frac{1}{2}} dy = \tfrac{3}{4} a^2 . \tfrac{1}{4}\pi a^2$$

$$= \frac{3\pi}{16} a^4;$$

hence $\Sigma(hAz) = \dfrac{\pi}{16} ac^3$.

Again, integrating from $y = 0$ to $y = (a^2 - x^2)^{\frac{1}{2}}$, and from $x = 0$ to $x = a$, we have

$$\Sigma(hC) = \iint z\,dx\,dy = \frac{c}{a}\iint (a^2 - x^2 - y^2)^{\frac{1}{2}} dx\,dy$$

$$= \frac{c}{a}\int \tfrac{1}{4}\pi (a^2 - x^2)\,dx = \tfrac{1}{6}\pi a^2 c.$$

Also

$$\Sigma(hCx) = \iint xz\,dx\,dy$$

$$= \frac{c}{a}\iint x(a^2 - x^2 - y^2)^{\frac{1}{2}} dx\,dy$$

$$= \frac{c}{a}\int_0^a \tfrac{1}{4}\pi (a^2 - x^2)\,x\,dx$$

$$= \tfrac{1}{16}\pi a^3 c.$$

Hence, R denoting the magnitude of the resultant pressure, we have, since by symmetry $\Sigma(hB) = \Sigma(hA)$,

$$R = g\rho\left\{\tfrac{2}{9}a^2c^4 + \tfrac{1}{36}\pi^2 a^4 c^2\right\}^{\frac{1}{2}} = \tfrac{1}{6} g\rho a c\left\{8c^2 + \pi^2 a^2\right\}^{\frac{1}{2}}.$$

Also, from the equation

$$\bar{z}\Sigma(hA) - \bar{x}\Sigma(hC) = \Sigma(hAz) - \Sigma(hCx),$$

we have $\quad \bar{z} . \tfrac{1}{3}ac^2 - \bar{x} . \tfrac{1}{6}\pi a^2 c = \tfrac{1}{16}\pi a c^3 - \tfrac{1}{16}\pi a^3 c,$

or $\quad 8\pi a\bar{x} - 16c\bar{z} = 3\pi(a^2 - c^2) \ldots\ldots\ldots\ldots (1).$

In like manner, from the equation

$$\bar{y}\Sigma(hC) - \bar{z} . \Sigma(hB) = \Sigma(hCy) - \Sigma(hBz),$$

as is evident from symmetry, we should obtain

$$8\pi a\bar{y} - 16c\bar{z} = 3\pi(a^2 - c^2) \ldots\ldots\ldots\ldots (2).$$

Again, it is manifest from symmetry that

$$\Sigma(hB) = \Sigma(hA), \quad \Sigma(hBx) = \Sigma(hAy):$$

hence from the equation

$$\bar{x}\Sigma(hB) - \bar{y}\Sigma(hA) = \Sigma(hBx) - \Sigma(hAy),$$

we have $$\bar{x} = \bar{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Any one of the equations (1), (2), (3), is deducible from the other two, any two of them being the equations to the direction of the resultant pressure.

3. A space bounded by the three coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1,$$

is filled with an elastic fluid without weight: to investigate the equations to the direction of the resultant pressure.

Since the unit of pressure will be constant, the three equations for the direction of the resultant reduce themselves to

$$\bar{y}\Sigma(C) - \bar{z}\Sigma(B) = \Sigma(Cy) - \Sigma(Bz) \ldots\ldots\ldots (1),$$

$$\bar{z}\Sigma(A) - \bar{x}\Sigma(C) = \Sigma(Az) - \Sigma(Cx) \ldots\ldots\ldots (2),$$

$$\bar{x}\Sigma(B) - \bar{y}\Sigma(A) = \Sigma(Bx) - \Sigma(Ay) \ldots\ldots\ldots (3).$$

Now, integrating first from $y = 0$ to $y = y$, and then from $z = 0$ to $z = c$,

$$\Sigma(A) = \iint dy\,dz = \int y\,dz$$

$$= b\int\left(1 - 2\frac{z^{\frac{1}{2}}}{c^{\frac{1}{2}}} + \frac{z}{c}\right)dz = \tfrac{1}{6}bc,$$

and $$\Sigma(Az) = \iint z\,dy\,dz = \int yz\,dz = b\int\left(1 - 2\frac{z^{\frac{1}{2}}}{c^{\frac{1}{2}}} + \frac{z}{c}\right)z\,dz$$

$$= \tfrac{1}{30}bc^2.$$

The values of the other analogous quantities are obvious from symmetry.

Hence, from (1),

$$\bar{y}\,.\,\tfrac{1}{6}ab - \bar{z}\,.\,\tfrac{1}{6}ca = \tfrac{1}{30}ab^2 - \tfrac{1}{30}ac^2,$$

or $$b(\bar{y} - \tfrac{1}{5}b) = c(\bar{z} - \tfrac{1}{5}c) \quad \ldots\ldots\ldots\ldots (4).$$

Similarly, from (2) and (3),

$$c(\bar{z} - \tfrac{1}{5}c) = a(\bar{x} - \tfrac{1}{5}a) \ldots\ldots\ldots\ldots (5),$$

$$a(\bar{x} - \tfrac{1}{5}a) = b(\bar{y} - \tfrac{1}{5}b) \ldots\ldots\ldots\ldots (6).$$

Any one of the three equations (4), (5), (6), may be deduced from the other two, and therefore any two of them determine the direction of the resultant pressure.

4. A light vessel in the form of a segment of an ellipsoid, the ends of which are sections made by a principal plane and a plane parallel to it, stands with its larger end on a horizontal plane: to determine how much fluid may be poured into it without its raising the vessel and forcing its way out.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{.................(1)},$$

the base of the segment being principal plane coinciding with the plane x, y.

The semi-axes of a section of the ellipsoid made by a plane parallel to the plane xy and at a distance z from it, will be equal to

$$a\left(1 - \frac{z^2}{c^2}\right)^{\frac{1}{2}}, \quad b\left(1 - \frac{z^2}{c^2}\right)^{\frac{1}{2}},$$

and therefore the area of this section will be equal to

$$\pi ab\left(1 - \frac{z^2}{c^2}\right).$$

From this it is plain that, h being the depth of the fluid, the volume of the fluid will be equal to

$$\int_0^h \pi ab\left(1 - \frac{z^2}{c^2}\right) dz = \pi ab\left(h - \frac{h^3}{3c^2}\right).$$

But the volume of a cylinder on the base πab of the ellipsoidal segment and of an altitude h, is equal to πabh. Hence the vertical pressure of the fluid on the vessel will be equal to the weight of a mass of the fluid, the volume of which is equal to

$$\pi abh - \pi ab\left(h - \frac{h^3}{3c^2}\right) = \frac{\pi abh^3}{3c^2},$$

and therefore, if W denote the weight of the vessel and ρ the density of the fluid, the greatest value of h, in order that the vessel may not be raised upwards, is given by the equation

$$W = \frac{\pi\rho gabh^3}{3c^2},$$

or $$h = \left(\frac{3c^2 W}{\pi\rho gab}\right)^{\frac{1}{3}}.$$

5. A thin hemispherical bowl is placed in vacuum and filled with fluid, the surface of which is subjected to a pressure equal to that of the atmosphere. If the bowl be divided into two equal parts by a vertical plane through the centre, and be kept together by a string in the diameter perpendicular to the plane of section, to find the tension of the string, the two halves of the bowl being supposed to be joined together at the lowest point.

Let AOB (fig. 9) be the dividing plane, C being the centre of the bowl, and O its lowest point. Let P be any point in the area of the bowl: join CP, and let OPQ be a quadrant of a great circle of the sphere: join CQ. Let r = the radius of the bowl, $\angle ACQ = \phi$, $\angle PCQ = \theta$, and h = the altitude of a column of the fluid of which the pressure is equal to that of the atmosphere. Then, T denoting the tension of the string, the moment of T about a tangent line at O parallel to AB, must be equal to the moment of the pressure of the fluid upon the surface $AOBQ$ about the same line.

Now the component of the pressure upon an elemental area $r \cos\theta d\phi . rd\theta$ at P, at right angles to the plane AOB, will act through the point C, and will be equal to

$$g\rho (h + r \sin\theta) . r \cos\theta d\phi . rd\theta . \cos\theta \sin\phi$$
$$= g\rho r^2 (h + r \sin\theta) \cos^2\theta \sin\phi d\theta d\phi ;$$

and therefore the moment of the whole pressure about the tangent line at O will be equal to

$$g\rho r^3 \int_0^\pi \int_0^{\frac{1}{2}\pi} (h + r \sin\theta) \cos^2\theta \sin\phi d\theta d\phi$$
$$= 2g\rho r^3 \int_0^{\frac{1}{2}\pi} (h + r \sin\theta) \cos^2\theta d\theta$$
$$= g\rho r^3 \int_0^{\frac{1}{2}\pi} \{h(1 + \cos 2\theta) + 2r \sin\theta \cos^2\theta\} d\theta$$
$$= g\rho r^3 \left(\frac{\pi}{2} h + \tfrac{2}{3} r\right).$$

Hence, Tr being the moment of T about the line at O, we see that

$$T = \tfrac{1}{6} g\rho r^2 (3\pi h + 4r).$$

6. A hemisphere, filled with fluid, is divided into four unconnected portions by two vertical planes through its axis at right angles to each other; the parts are just kept together by four strings fastened at the centre and at four points in the surface: to find the position and tension of one of the strings.

Let the equation to the hemisphere be

$$x^2 + y^2 + z^2 = r^2,$$

the planes of x, z, and y, z, being the two dividing planes. Then, T being the tension of the string in the octant of space $+x$, $+y$, $+z$, we shall have, ρ denoting the density of the fluid,

$$T = \tfrac{1}{6} g\rho r^3 (\pi^2 + 8)^{\frac{1}{2}},$$

the equations to the direction of T being

$$y = \frac{2}{\pi} z = x.$$

7. A sphere is just filled with a homogeneous fluid; to determine the resultant of the pressures upon either of the hemispheres into which it is divided by a vertical plane.

If the equation to the sphere be

$$x^2 + y^2 + z^2 = r^2,$$

the plane of yz being the dividing plane, and the axis of z extending vertically downwards; then, ρ being the density of the fluid and R the resultant pressure,

$$R = \tfrac{5}{6} g\rho\pi r^3,$$

and the equations to the direction of R will be

$$y = 0, \quad 3z = 4x.$$

8. To find the form of an open vessel full of fluid, supposed to be a surface of revolution with its axis vertical, that the whole horizontal pressure exerted upon it by the fluid may be the greatest possible; the altitude and volume of the vessel being given.

The vessel will be a cone, and, if h be its altitude and c^3 its volume, the radius of its base will be equal to

$$\left(\frac{3c^3}{\pi h}\right)^{\frac{1}{2}}.$$

9. A vessel in the form of a paraboloid of revolution is partly filled with fluid, and then inverted upon a horizontal plane: having given the weight of the vessel, to find the altitude of the greatest volume of fluid which can be contained without running out at the rim of the vessel.

If W denote the weight of the vessel, l the latus rectum of the paraboloid, and x the required altitude,

$$x = \frac{2W}{\pi g\rho l}.$$

10. A hollow vessel in the form of a tetrahedron is filled with a known fluid, and placed in an inverted position on a horizontal plane: to find the least weight of the vessel that the fluid may not escape from under it.

Least weight of the vessel = twice the weight of the fluid.

CHAPTER IV.

EQUILIBRIUM OF SOLID BODIES FLOATING FREELY.

In order that a solid, partially or totally immersed in a fluid, may float in equilibrium, it is sufficient and necessary that the resultant pressure of the fluid upon it act in the vertical line passing through its centre of gravity and be equal to its weight. If the fluid be incompressible (or even if it be elastic, provided that the solid be totally immersed), these conditions are equivalent to the two following conditions, viz. that the weight of the fluid displaced be equal to that of the solid, and that the line joining the centres of gravity of the solid and fluid displaced be vertical.

The discovery of the principles of the equilibrium of floating bodies is due to Archimedes; his investigations on this subject were published in his treatise entitled Περὶ τῶν ὀχουμένων, the original of which is no longer extant. The Latin translation by Tartaglia of this remarkable work, by which our knowledge of the hydrostatical discoveries of Archimedes has been preserved, is entitled *De iis quæ vehuntur in aquâ;* an edition of which by Commandine was published at Bologna in the year 1565.

Section I.

Bodies floating in Incompressible Homogeneous Fluids.

1. To find the positions in which a square lamina $ABCD$ (fig. 10) may float in a fluid with one angle A out of the fluid, the plane of the lamina being vertical.

Let PQ be the line of floatation: let $AP = x$, $AQ = y$. Join AC and bisect it in G. Bisect PQ in F, join AF, and take $HF = \frac{1}{3}AF$. Then G and H will be the centres of gravity

of $ABCD$ and APQ respectively. Join GH and draw FE parallel to HG. Then, in order that the lamina may be at rest, G and the centre of gravity of $PBCDQ$ must be in the same vertical line: but H, G, and the centre of gravity of $PBCDQ$ are in a straight line; hence GH and EF must be vertical lines. Join PE, QE; these two lines will be equal, because $PF = QF$ and $\angle PFE = \angle QFE$.

Let a denote the length of a side of the square, ρ the density of the lamina, and σ of the fluid.

Then, the weight of the displaced fluid being equal to that of the lamina, we must have

$$\sigma . PBCDQ = \rho . a^2,$$

or
$$\sigma (a^2 - \tfrac{1}{2}xy) = \rho a^2,$$

and therefore
$$xy = 2\left(1 - \frac{\rho}{\sigma}\right) a^2 \ldots\ldots\ldots\ldots\ldots (1).$$

Again,
$$PE^2 = x^2 + AE^2 - 2x\, AE \cos\frac{\pi}{4},$$

$$QE^2 = y^2 + AE^2 - 2y\, AE \cos\frac{\pi}{4},$$

and therefore, since $PE = QE$,

$$(x - y)\left\{x + y - 2\, AE \cos\frac{\pi}{4}\right\} = 0:$$

but
$$AE : AG :: AF : AH :: 3 : 2,$$

$$AE = \tfrac{3}{2} AG,$$

$$2\, AE \cos\frac{\pi}{4} = 3\, AG \cos\frac{\pi}{4} = \tfrac{3}{2} a:$$

hence
$$(x - y) . (x + y - \tfrac{3}{2} a) = 0,$$

which equation gives us either

$$x - y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or
$$x + y - \tfrac{3}{2} a = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

Taking (1) and (2) we have

$$x^2 = 2\left(1 - \frac{\rho}{\sigma}\right) a^2 = y^2,$$

which gives one position of equilibrium, provided that $\frac{\rho}{\sigma}$ be less than 1 and greater than $\frac{1}{2}$.

Taking (1) and (3) we have

$$(x-y)^2 = a^2\left(\frac{8\rho}{\sigma} - \frac{23}{4}\right),$$

or

$$x - y = \pm\, a\left(\frac{8\rho}{\sigma} - \frac{23}{4}\right)^{\frac{1}{2}}$$

and

$$x + y = \tfrac{3}{2}a,$$

and therefore

$$x = \tfrac{1}{4}a\left\{3 \pm \left(32\frac{\rho}{\sigma} - 23\right)^{\frac{1}{2}}\right\},$$

$$y = \tfrac{1}{4}a\left\{3 \mp \left(32\frac{\rho}{\sigma} - 23\right)^{\frac{1}{2}}\right\}.$$

Thus there will be two other positions of equilibrium, provided that x and y be both possible and positive quantities less than a. This will be the case if the two inequalites hold good, viz.

$$\frac{\rho}{\sigma} > \frac{23}{32}, \quad \left(32\frac{\rho}{\sigma} - 23\right)^{\frac{1}{2}} < 1,$$

that is, if $\frac{\rho}{\sigma}$ be greater than $\frac{23}{32}$ and less than $\frac{24}{32}$.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 165.

2. A given triangular lamina ABC (fig. 11), all the sides of which are unequal, floats in a fluid with one angle (A) immersed; to determine its positions of equilibrium, the plane of the lamina being vertical.

Let D be the middle point of BC, and E that of $B'C'$, $B'C'$ being the line of floatation. Join AD, AE, DE, $B'D$, $C'D$. Then, G being the centre of gravity of the lamina and H of the fluid displaced, $AG = \frac{2}{3}AD$, $AH = \frac{2}{3}AE$, and therefore

$$AG : AD :: AH : AE;$$

which shews that GH is parallel to DE. But GH must be vertical and therefore at right angles to $B'C'$: hence DE is at right angles to $B'C'$. Since $\angle\, DEB' = \angle\, DEC'$, and $B'E = C'E$, it follows that $DB' = DC'$.

Let $AB' = x$, $AC' = y$, $AD = l$, $AC = b$, $AB = a$, $\angle\, B'AD = \lambda$, $\angle\, C'AD = \mu$: also let ρ = the density of the solid, and σ = that of the fluid. Then, the weight of BAC being equal to that of the fluid displaced by $B'AC'$, we have

$$\tfrac{1}{2}xy\,\sigma \sin\angle\, B'AC' = \tfrac{1}{2}ab\rho \sin\angle\, BAC,$$

or $$\sigma xy = \rho ab \quad \text{.................(1)}.$$

Again $$B'D^2 = x^2 + l^2 - 2lx \cos\lambda,$$
$$C'D^2 = y^2 + l^2 - 2ly \cos\mu;$$

and therefore, $B'D$ being equal to $C'D$,

$$x^2 + l^2 - 2lx\cos\lambda = y^2 + l^2 - 2ly\cos\mu,$$

or $$(x - l\cos\lambda)^2 - (y - l\cos\mu)^2 = l^2(\cos^2\lambda - \cos^2\mu)\text{...(2)}.$$

The values of x and y deducible from the two equations (1) and (2) will coincide with the coordinates of the intersections of two equilateral hyperbolæ of which (1) and (2) are taken as the equations.

From the construction of these two curves it is easily seen that they can intersect each other either once only or three times only in the positive quadrant: there may therefore be one and there may be three positions of equilibrium. Should all the points of intersection of the hyperbolæ be such that x is greater than a or y than b, the lamina will not rest in any position with only one angle immersed.

Cor. 1. If the triangle be isosceles, A being the vertical angle, $\cos\lambda = \cos\mu$, and then, instead of the equation (2), we shall have $$(x - y)\{x + y - 2l\cos\lambda\} = 0,$$

or $$x - y = 0, \quad x + y = 2l\cos\lambda,$$

the equations to two straight lines instead of an hyperbola: these two lines may cut the hyperbola denoted by the equation (1) in three points in the positive quadrant; the former straight line will certainly cut it in one. Thus there may be one and there may be three positions of equilibrium.

Cor. 2. If the triangle be equilateral,
$$2l\cos\lambda = \tfrac{3}{2}a,$$

and therefore, instead of the equation (2), we have either of the two following, $x - y = 0$, or $x + y = \tfrac{3}{2}a$.

From (1) and the former of these we get

$$x^2 = \frac{\rho}{\sigma}a^2,$$

which gives one position of equilibrium.

From (1) and the latter equation

$$(x - y)^2 = \tfrac{9}{4}a^2 - 4\frac{\rho}{\sigma}a^2$$

$$= \tfrac{1}{4}a^2\left(9 - 16\frac{\rho}{\sigma}\right),$$

$$x - y = \pm\tfrac{1}{2}a\left(9 - 16\frac{\rho}{\sigma}\right)^{\frac{1}{2}},$$

$$x + y = \tfrac{3}{2}a:$$

whence
$$x = \tfrac{1}{4}a\left\{3 \pm \left(9 - 16\frac{\rho}{\sigma}\right)^{\frac{1}{2}}\right\},$$

$$y = \tfrac{1}{4}a\left\{3 \mp \left(9 - 16\frac{\rho}{\sigma}\right)^{\frac{1}{2}}\right\}.$$

That these results may be admissible, x and y must be both possible, both positive, and both less than a. Hence we must have

$$9 > 16\frac{\rho}{\sigma}, \quad \text{and} \quad \left(9 - 16\frac{\rho}{\sigma}\right)^{\frac{1}{2}} < 1,$$

that is, $\frac{\rho}{\sigma}$ must be less than $\frac{9}{16}$ and greater than $\frac{8}{16}$.

Herman: *Phoronomia*, p. 160.
Bossut: *Traité d'Hydrodynamique*, tom. I. p. 155.
Poisson: *Traité de Mécanique*, tom. II. p. 580.

3. A given triangular lamina ABC, all the sides of which are unequal, floats in a fluid with two angles B and C immersed; to determine its position of equilibrium, the plane of the lamina being vertical.

Adhering to the diagram and notation of the preceding problem, it is plain that, $BB'C'C$ being in the fluid and $B'AC'$ being in vacuum,

$$BB'C'C \times \sigma = BAC \,.\, \rho,$$

whence
$$\sigma(ab - xy) = \rho ab,$$

or
$$\sigma xy = (\sigma - \rho)\,ab.$$

Also the centres of gravity of $BB'C'C$ and BAC, and therefore of $B'AC'$, must be in the same vertical line.

Thus the solution of this problem coincides in form with that of the preceding, $\sigma - \rho$ replacing ρ throughout: and therefore we know that, supposing the angles B and C to be both immersed, there may be one and there may be three positions of equilibrium.

COR. 1. There will be therefore at most three positions of equilibrium when A alone is immersed, and three when A alone is out of the fluid: thus, varying the angles of the triangle, we see that, at most, the triangle can have only eighteen positions of equilibrium.

COR. 2. There will be three positions of equilibrium, A being without the fluid, if the triangle be equilateral, provided that $\frac{\rho}{\sigma}$ be greater than $\frac{7}{16}$, and less than $\frac{8}{16}$.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 160.
Poisson: *Traité de Mécanique*, tom. II. p. 584.

4. To find the positions of equilibrium of a parabolic lamina BAC, bounded by a double ordinate BC at right angles to its axis, which floats in a fluid with its vertex A immersed, the plane of the lamina being vertical.

First, let us suppose the axis to be vertical, and let c be that portion of the axis which is within the fluid, a being the length of the axis of the parabolic area and $2b$ its double ordinate. Let ρ be the density of the lamina and σ of the fluid.

Then, for equilibrium,

$$\tfrac{2}{3}ab \, . \, \rho = \tfrac{2}{3}c \, . \, b\left(\frac{c}{a}\right)^{\frac{1}{2}} . \, \sigma,$$

$$\rho a^{\frac{3}{2}} = \sigma c^{\frac{3}{2}},$$

which determines c, and therefore the position of equilibrium.

Next, let us suppose the parabola to lie in an oblique position. Let Qq (fig. 12) denote the line of floatation, G the centre of gravity of the lamina, which will lie in the axis AD; H the centre of gravity of the fluid displaced, which will lie in a diameter VP through a point P, the tangent of which is horizontal. In order that the lamina may be at rest, HG must be vertical.

Draw PE vertical to meet AD in E, and draw PM at right angles to AD. Let the tangent at P meet the axis DA produced in the point T. Let $\angle PTA = \theta$, and $4m$ = the latus rectum.

Then, by the nature of a parabola,

$$AM = m \cot^2 \theta, \qquad ME = 2m;$$

and
$$GE = PH = \tfrac{3}{5} PV, \qquad AG = \tfrac{3}{5} AD.$$

Hence
$$m \cot^2 \theta + 2m + \tfrac{3}{5} PV = \tfrac{3}{5} a,$$

or, since $b^2 = 4ma$,

$$\frac{b^2}{4a} \cot^2 \theta + \frac{b^2}{2a} + \tfrac{3}{5} PV = \tfrac{3}{5} a \ldots\ldots\ldots\ldots (1).$$

Again,
$$\rho \,.\, \text{area } BAC = \sigma \,.\, \text{area } QPq,$$

or
$$\tfrac{2}{3}\rho \,.\, ab = \tfrac{4}{3}\sigma \,.\, QV.PV.\sin\theta,$$

$$\rho^2 a^2 b^2 = \sigma^2 \,.\, QV^2 \,.\, PV^2 \,.\, \sin^2\theta$$

$$= \sigma^2 \,.\, 4SP \,.\, PV^3 \,.\, \sin^2\theta,$$

S being the focus;

but
$$SP = \frac{2m}{1 - \cos 2\theta} = \frac{m}{\sin^2\theta};$$

hence
$$\rho^2 a^2 b^2 = 4\sigma^2 \,.\, PV^3 \,.\, m,$$

$$= \sigma^2 \,.\, PV^3 \,.\, \frac{b^2}{a},$$

$$PV = a\left(\frac{\rho}{\sigma}\right)^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2),

$$\cot^2\theta = \frac{12a^2}{5b^2}\left\{1 - \left(\frac{\rho}{\sigma}\right)^{\frac{2}{3}}\right\} - 2 \ldots\ldots\ldots\ldots (3).$$

From (2) and (3) we know PV and θ, which will have two equal values with opposite signs. The positions of equilibrium of the lamina are therefore determined. The values of θ will be impossible if

$$\frac{b^2}{a^2} > \tfrac{6}{5}\left\{1 - \left(\frac{\rho}{\sigma}\right)^{\frac{2}{3}}\right\}.$$

Thus the parabola will sometimes have one and sometimes three positions of rest.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 170.

5. A solid homogeneous cube floats in a fluid with only one angle immersed; to determine its position of equilibrium.

Let OA, OB, OC, (fig. 13) be three edges of the cube, O being the angle immersed. Let PQR be the plane of floatation. Let $OP = a$, $OQ = b$, $OR = c$, and let 2α denote the length of an edge of the cube. The equation to the plane PQR, OA, OB, OC, being taken as the axes of x, y, z, respectively, will be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Also, $\bar{x}, \bar{y}, \bar{z}$, being the coordinates of the centre of gravity of the pyramid $OPQR$, and $\bar{x}', \bar{y}', \bar{z}'$, of the centre of gravity of the cube.

$$\bar{x} = \tfrac{1}{4}a, \quad \bar{y} = \tfrac{1}{4}b, \quad \bar{z} = \tfrac{1}{4}c,$$

$$\bar{x}' = \alpha, \quad \bar{y}' = \alpha, \quad \bar{z}' = \alpha.$$

But the line passing through the two points $(\bar{x}, \bar{y}\ \bar{z})$ and $(\bar{x}', \bar{y}', \bar{z}')$ must be at right angles to the plane PQR; hence

$$a(\alpha - \tfrac{1}{4}a) = b(\alpha - \tfrac{1}{4}b) = c(\alpha - \tfrac{1}{4}c),$$

which equations are equivalent to any two of the equations

$$\alpha(a - b) = \tfrac{1}{4}(a^2 - b^2),$$

$$\alpha(b - c) = \tfrac{1}{4}(b^2 - c^2),$$

$$\alpha(c - a) = \tfrac{1}{4}(c^2 - a^2).$$

Now from the first of these three equations we have

$$a - b = 0,$$

or

$$4\alpha = a + b:$$

the latter of these results is incompatible with the condition that the angle O is the only one immersed. Hence we must have $a = b$, and similarly, from the second and third equations, $b = c$, $c = a$. Thus it appears that the cube can float in only one way, provided that only one angle be immersed.

Again, ρ being the density of the cube, and σ of the fluid, the weight of the fluid pyramid $OPQR$ will be $\frac{1}{6}abc.\sigma.g$, and that of the cube will be $8a^3.\rho.g$. Hence

$$\tfrac{1}{6}abc.\sigma.g = 8a^3.\rho.g,$$

or

$$\sigma abc = 48\rho a^3,$$

But $a = b = c$; hence

$$\sigma a^3 = 48\rho a^3.$$

$$c = b = a = 2a\left(\frac{6\rho}{\sigma}\right)^{\frac{1}{3}}.$$

In order that the problem may be possible, a, b, c, must be less than $2a$; hence we must have the inequality

$$\left(\frac{6\rho}{\sigma}\right)^{\frac{1}{3}} < 1,$$

or

$$\frac{\rho}{\sigma} < \tfrac{1}{6}.$$

6. A hemispherical bowl of given weight ($2W$) floats upon a fluid with one third of its axis below the surface: if a weight $5W$ be put into it, how much of its axis will be immersed when it is in its position of rest?

Length of the axis immersed $= \frac{2}{3} \times$ length of the whole axis.

7. A cylindrical vessel in an upright position contains a quantity of fluid: supposing a solid cylinder of smaller radius to be placed in the fluid so as to float at rest in an upright position, to determine how much the fluid will rise in the hollow cylinder.

Let ρ denote the density of the solid cylinder and σ of the fluid; let R be the radius of the internal surface of the hollow cylinder, r the radius and h the altitude of the solid cylinder. Then, x denoting the altitude through which the fluid rises above its original surface,

$$x = \frac{\rho r^2}{\sigma R^2}\,h.$$

8. A hollow cone, of which the vertical angle is $\frac{1}{3}\pi$, is placed with its axis vertical and vertex downwards: to find what quantity of fluid it must contain, in order that a given sphere,

the density of which is half that of the fluid, may sink just deep enough to touch the surface of the cone.

The altitude of the surface of the fluid above the vertex of the cone, when the sphere is immersed, must be equal to the diameter of the sphere.

9. To determine the depth to which a given paraboloid must be immersed with its axis vertical and vertex downwards in a fluid of three times the density of itself, in order that it may remain there at rest.

If a denote the length of the axis of the paraboloid, and x the required depth,

$$x = \frac{a}{\sqrt{3}}.$$

10. To determine the density of a square lamina which floats in a given fluid, with one angle in the surface and two below the surface.

If ρ be the density of the lamina, and σ of the fluid, then

$$\rho : \sigma :: 3 : 4.$$

11. To determine the thickness of a hollow sphere of given material and external superficies, which just floats in a given fluid.

Let c^2 represent the area of the superficies of the sphere, ρ the density of the sphere and ρ' of the fluid; then the required thickness will be equal to

$$\frac{c}{2\pi^{\frac{1}{2}}}\left\{1 - \left(\frac{\rho - \rho'}{\rho}\right)^{\frac{1}{3}}\right\}.$$

12. To find the positions of equilibrium of a parabolic lamina bounded by a double ordinate BC at right angles to its axis, which floats in a fluid with its vertex A immersed, the plane of the lamina being vertical, and its centre of gravity G not coinciding with the centre of gravity of its figure.

Let AD (fig. 14) be the axis of the parabola, Qq the line of floatation in an oblique position of equilibrium, P the point of contact of a tangent to the parabola parallel to Qq, PV a line drawn parallel to AD to intersect Qq in the point V, G the

centre of gravity of the lamina, GF a line perpendicular to AD and meeting it in F, PM a perpendicular to AD. Then, l denoting the latus rectum, ρ the density of a homogeneous mass of the same volume and weight as the parabolic lamina, and σ the density of the fluid, we have, for the determination of the positions of equilibrium, putting $AD = a$, $AF = h$, $FG = k$, $PV = x$, $PM = z$,

$$x = \left(\frac{\rho}{\sigma}\right)^{\frac{2}{3}} a,$$

and
$$10z^3 - (10lh - 6lx - 5l^2)\,z - 5l^2k = 0.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 173.

13. Three spheres, of equal volumes but different densities, float in a fluid, touching each other; to find the inclination of their common tangent plane to the surface of the fluid.

If θ be the required angle of inclination, r the radius of each sphere, and c, c', c'', the altitudes of the centres of the three spheres above the surface,

$$3r^2 \sin^2\theta = c(c - c') + c'(c' - c'') + c''(c'' - c),$$

the values of c, c', c'', being determined by the equations

$$\sigma(r - c)^2(2r + c) = 4r^3\rho, \quad \sigma(r - c')^2(2r + c') = 4r^3\rho',$$
$$\sigma(r - c'')^2(2r + c'') = 4r^3\rho'',$$

σ being the density of the fluid, and ρ, ρ', ρ'', the densities of the three spheres.

SECTION II.

Bodies floating in Incompressible Heterogeneous Fluids.

1. A uniform solid cylinder rests, with its axis vertical, in two fluids which do not mix. The densities of the cylinder, the upper and the lower fluids, are as 2 : 3 : 5 respectively, and the depth of the rarer fluid is equal to one-fourth of the axis of the cylinder: to determine the position of the centre of gravity of the cylinder in relation to the surface of the upper fluid.

Let the three densities be 2ρ, 3ρ, 5ρ; h the whole length of the cylinder, x the depth to which it is immersed in the lower fluid, and K the area of its section.

Then, the weight of the cylinder being equal to that of the fluid displaced,

$$2\rho Kgh = 3\rho Kg \cdot \tfrac{1}{4}h + 5\rho Kgx,$$

$$2h = \tfrac{3}{4}h + 5x$$

$$5x = \tfrac{5}{4}h, \quad x = \tfrac{1}{4}h.$$

This result shews that half the axis of the cylinder is immersed in fluid, or that its middle point is in the surface of the upper fluid.

2. A circular lamina rests with a diameter in the surface of a fluid, the density of which varies as the depth: to compare the density of the solid with the density of the fluid at the lowest point of the lamina.

Let a be the radius of the circle, k the density of the fluid at a unit of depth, τ the thickness of the lamina, r the distance of any point of the lamina from the centre of the circle, and θ the inclination of r to the vertical.

Then the weight of the fluid displaced is equal to

$$g\tau \int\int r\,d\theta\,dr \,.\, kr\cos\theta$$

$$= k\tau g \int_0^a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} r^2 \cos\theta\, d\theta\, dr$$

$$= \tfrac{1}{3}k\tau g a^3 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos\theta\, d\theta$$

$$= \tfrac{2}{3}k\tau g a^3 = \tfrac{2}{3}g\sigma\tau a^2,$$

σ being the density of the fluid at the depth a.

Also the weight of the lamina is equal to

$$\tfrac{1}{2}\pi a^2 g\rho\tau.$$

Hence

$$\tfrac{2}{3}g\sigma\tau a^2 = \tfrac{1}{2}\pi g\rho\tau a^2,$$

$$\tfrac{2}{3}\sigma = \tfrac{1}{2}\rho\pi,$$

or

$$\rho : \sigma :: 4 : 3\pi.$$

3. A cylinder, with its axis vertical, floats in two fluids of different densities, the upper end of the cylinder coinciding with

the surface of the upper fluid: to compare the parts into which the cylinder is divided by their common surface.

If x denote the length of the upper and y of the lower portion of the cylinder, σ the density of the upper and σ' of the lower fluid, and ρ the density of the cylinder,

$$x : y :: \sigma' - \rho : \rho - \sigma.$$

4. To determine the magnitude of a sphere of given density which will rest just immersed in a fluid of which the density varies as the depth.

If ρ denote the density of the sphere, σ the density of the fluid at a depth c below the surface, then, the radius of the sphere being denoted by r,

$$r = \frac{c\rho}{\sigma}.$$

5. A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid: to find how much its density must be increased, that it may rest with its base in the common surface of the fluids.

If σ be the density of the upper and σ' of the lower fluid; ρ that of the cone in the former and ρ' in its latter position; h the length of the whole axis of the cone, and h' of that portion of it which, in the former position of the cone, is immersed in the lower fluid,

$$\rho' - \rho = \frac{h^3 - h'^3}{h^3}(\sigma' - \sigma).$$

6. The density of a fluid varies as a certain power of the depth: to find this power when a solid sphere, the density of which is one-fifth of that of the fluid at a depth equal to the diameter of the sphere, will rest just totally immersed in the fluid.

The required power is 3 or -8.

SECTION III.

Bodies floating under the action of Elastic Fluids.

1. A solid cylinder AB (fig. 15), with its axis vertical, just passes through a smooth circular aperture C in the upper end of a vertical cylindrical box HK containing an elastic fluid. Supposing the elasticity of the fluid, when it fills the whole box, to be given, to determine the length of the immersed portion of the solid cylinder that it may be in equilibrium.

Let r, R, denote the radii of the solid and hollow cylinder respectively. Supposing the total weight of the fluid to be inconsiderable, let p represent the unit of pressure at every point of the fluid before and p' after the insertion of the solid cylinder.

Then, the pressure on the base of the solid cylinder being equal to $\pi r^2 p'$, we shall have, W denoting the weight of the solid cylinder,
$$\pi r^2 p' = W \dots\dots\dots\dots\dots\dots (1).$$

Let h denote the height of the hollow cylinder, and x the length of the immersed portion of the solid one; then the volume of the fluid, corresponding to the elasticity p, being $\pi R^2 h$, and, corresponding to the elasticity p', $\pi R^2 h - \pi r^2 x$, we shall have
$$p' . (\pi R^2 h - \pi r^2 x) = p . \pi R^2 h,$$
and therefore, from (1),
$$W(R^2 h - r^2 x) = \pi R^2 r^2 p h,$$
$$x = \frac{R^2 h}{r^2 W}(W - \pi r^2 p).$$

2. A solid cylinder, the axis of which is vertical, floats in an incompressible fluid; the fluid is contained in a vessel enclosing air, into which a piston fits; supposing the density of the air to be changed by a change in the position of the piston, to ascertain the alteration in the position of equilibrium of the cylinder.

Let h be the altitude of the cylinder, A the area of its base, and x the depth to which it is immersed in the incompressible fluid. Let ρ, σ, ρ', be the respective densities of the cylinder,

the incompressible fluid, and the enclosed air. Then, the weight of the cylinder being $g\rho hA$, the weight of the displaced portion of the incompressible fluid $g\sigma xA$, and, supposing the density of the enclosed air to be the same throughout, the weight of the displaced air $g\rho'(h-x)A$, we have the equation

$$g\rho hA = g\sigma xA + g\rho'(h-x)A,$$

$$\rho h = \sigma x + \rho'(h-x),$$

$$(\rho-\rho')h = (\sigma-\rho')x,$$

$$x = h\frac{\rho-\rho'}{\sigma-\rho'}.$$

Since σ is greater than ρ, the coefficient of h in this expression will be a proper fraction. Hence, by increasing ρ', x will decrease, and, by diminishing ρ', x will increase; that is, the cylinder will rise or fall, accordingly as the piston is pushed inwards or pulled outwards.

3. A heavy piston rests in a close vertical cylinder, having equal quantities of gas above and below it: to determine its position of equilibrium, the radius of the external surface of the solid cylinder being equal to that of the internal surface of the hollow one.

Let C denote the mass of the gas in each of the compartments; x the length of the upper and y of the lower cylinders of gas, σ being the density of the former and σ' of the latter. Let h be the length of the solid cylinder, ρ its density, and A the area of one of its circular sections. Let H denote the length of the hollow cylinder; also let P, P', denote the pressures of the two volumes of gas upon the upper and lower ends of the solid cylinder respectively.

Then, for the equilibrium of the solid cylinder,

$$P + g\rho hA = P' \quad \ldots\ldots\ldots\ldots\ldots (1).$$

But, k denoting a constant quantity dependent upon the nature of the gas,

$$P = k\sigma A, \qquad P' = k\sigma' A:$$

hence, from (1),

$$k\sigma + g\rho h = k\sigma',$$

or, since $$C = A\sigma x = A\sigma' y,$$

$$kC.\left(\frac{1}{y} - \frac{1}{x}\right) = Ag\rho h = W \dots\dots\dots\dots (2),$$

W denoting the weight of the solid cylinder.

Again, it is plain that

$$x + y + h = H \dots\dots\dots\dots\dots\dots (3).$$

From the equations (2) and (3) the values of x and y may easily be determined.

In fact, by the solution of these equations, we shall obtain

$$x = \frac{H-h}{2} - \frac{kC}{W} + \left\{\frac{(H-h)^2}{4} + \frac{k^2C^2}{W^2}\right\}^{\frac{1}{2}};$$

$$y = \frac{H-h}{2} + \frac{kC}{W} - \left\{\frac{(H-h)^2}{4} + \frac{k^2C^2}{W^2}\right\}^{\frac{1}{2}}.$$

CHAPTER V.

EQUILIBRIUM OF SOLID BODIES FLOATING UNDER CONSTRAINT.

In order to ascertain the circumstances of the equilibrium of solid bodies, floating in fluids and acted upon by any forces of constraint, we must observe that the body is acted upon by the weight of the fluid displaced acting vertically upwards through the centre of gravity of this portion of the fluid, by its own weight acting vertically downwards through its own centre of gravity, and by the forces of constraint. The equations of condition, that the forces involved under these three heads may produce rest, will determine the required circumstances of equilibrium.

Section I.

Incompressible Homogeneous Fluids.

1. A uniform rod AB (fig. 16) turns about a smooth hinge at A at the bottom of a vessel containing fluid, the depth of which is equal to half AB, and of which the density is equal to three times that of the rod: to find the inclination of the rod to the vertical when there is equilibrium, and to determine what weight must be suspended at B in order to depress the rod through 15°.

Let κ be the area of a perpendicular section of the rod, a the length of the rod, x the length of the immersed portion AC, ρ the density of the rod and σ that of the fluid. Let b be the length of a portion of the rod the weight of which shall be equal to the weight suspended from B.

Then, in regard to the first part of the problem, we have, taking moments about A,

$$g\sigma\kappa x \,.\, \tfrac{1}{2}x \sin\theta = g\rho\kappa a \,.\, \tfrac{1}{2}a \sin\theta,$$

or $$\sigma x^2 = \rho a^2 :$$

but $$\sigma = 3\rho, \quad x \cos\theta = \tfrac{1}{2}a;$$

hence $$\tfrac{3}{4}a^2 = a^2 \cos^2\theta,$$

and therefore $$\cos\theta = \frac{\sqrt{3}}{2}, \quad \theta = \frac{\pi}{6}.$$

Next, the weight being suspended from B,

$$g\sigma\kappa x \,.\, \tfrac{1}{2}x \sin\theta = g\rho\kappa a \,.\, \tfrac{1}{2}a \sin\theta + g\rho\kappa b \,.\, a \sin\theta,$$

or $$\sigma x^2 = \rho a^2 + 2ab\rho;$$

but $\sigma = 3\rho$, and, θ being equal to $\frac{\pi}{4}$, $x = \frac{a}{\sqrt{2}}$; hence

$$\frac{3a^2}{2} = a^2 + 2ab,$$

whence $$b = \tfrac{1}{4}a,$$

the required weight being therefore $\tfrac{1}{4}g\rho\kappa a$.

2. A cone ABV, (fig. 17), the vertex V of which is fixed at the bottom of a vessel containing fluid, is in equilibrium with its slant side vertical and the lowest point A of its base just touching the surface of the fluid: to compare the density of the cone with that of the fluid.

Let O be the centre of the base AB of the cone, O' that of the elliptical section AC made by a plane coinciding with the surface of the fluid. Join VO, VO', and draw CD parallel to BA.

Let G, H, points in the lines VO, VO', be the centres of gravity of the cones AVB, AVC, VG being equal to $\tfrac{3}{4}OV$, and VH to $\tfrac{3}{4}O'V$.

Let $VO = h$, $VC = h'$, $AO = a$, $AO' = a'$, $\angle AVO = \alpha$, A = the area of the circle AB, A' = the area of the ellipse AC, ρ = the density of the cone, and σ = the density of the fluid. Then, the moment of the solid cone AVB about V being equal to $\rho \,.\, \tfrac{1}{3}hA \,.\, \tfrac{3}{4}h \sin\alpha$, and that of the fluid cone AVC being

$\sigma \,.\, \frac{1}{3} h' A' \,.\, \frac{3}{4} a'$, we have, since these two moments must be equal in order that the cone may be at rest,

$$\rho h^2 A \sin \alpha = \sigma h' a' A' \ldots\ldots\ldots\ldots\ldots (1).$$

But $\quad h = a \cot \alpha, \quad A = \pi a^2,$

$$a' = a \cos \alpha, \quad h' = 2a' \cot 2\alpha = 2a \cos \alpha \cot 2\alpha,$$

$$A' = \tfrac{1}{4}\pi \,.\, CA \,.\, (AB \,.\, CD)^{\frac{1}{2}} = \tfrac{1}{2}\pi a' \left(2a \,.\, 2a' \frac{\cos 2\alpha}{\cos \alpha}\right)^{\frac{1}{2}}$$

$$= \pi a' \left(a a' \frac{\cos 2\alpha}{\cos \alpha}\right)^{\frac{1}{2}}$$

$$= \pi a^2 \cos \alpha \, (\cos 2\alpha)^{\frac{1}{2}}.$$

Hence, substituting these values of h, A, a', h', A', in (1), we get

$$\pi a^4 \rho \frac{\cos^2 \alpha}{\sin \alpha} = 2\pi a^4 \sigma \cos^3 \alpha \frac{(\cos 2\alpha)^{\frac{3}{2}}}{\sin 2\alpha},$$

$$\rho = 2\sigma \sin \alpha \cos \alpha \frac{(\cos 2\alpha)^{\frac{3}{2}}}{\sin 2\alpha}$$

$$= \sigma \, (\cos 2\alpha)^{\frac{3}{2}}.$$

3. A body immersed in a fluid is balanced by a weight P to which it is attached by a string passing over a fixed pully; and, when half immersed, is balanced in the same manner by a weight $2P$: to compare the densities of the body and fluid.

Density of the body : Density of the fluid :: 3 : 2.

4. If a, b, c, be the depths at which the lower surface of a cylinder will float in equilibrium in a fluid, when attached to three weights P, Q, R, respectively, which are connected with the centre of its upper end by means of a string passing freely over a fixed pully, to determine the relation between a, b, c, P, Q, R.

The required relation is expressed by the equation

$$P(b - c) + Q(c - a) + R(a - b) = 0.$$

5. A rectangular lamina $ABCD$ (fig. 18) partially immersed in a fluid, rests vertically: one angle A is attached to a smooth

hinge, and two angles C and D are within the fluid; the angle at which the side AB of the rectangle is inclined to the surface, is equal to 45°: having given the density of the fluid and the ratio of AB to BC, to determine the excess of the density of the fluid above that of the lamina.

If σ denote the density of the fluid and m the ratio of AB to BC, the required excess is equal to

$$\frac{3\sigma}{m^2 + m}.$$

6. A uniform rectangular board $ABCD$ rests, with its lowest corner A immersed in a fluid, its highest C held up by a string, and the lowest B of the other two corners at the surface of the fluid: to determine the ratio between the density of the board and that of the fluid.

If ρ be the density of the board and σ that of the fluid, then

$$\frac{\rho}{\sigma} = \frac{AB}{BC}.$$

7. A uniform cylinder is moveable about a fixed horizontal axis, which is a tangent to the circumference of its upper end at the highest point: the vertical distance of the fixed axis from the surface of a fluid, in which part of the cylinder is immersed, is equal to the diameter of the base, the diameter being equal to one half of the axis of the cylinder; the cylinder rests with its base inclined at an angle of 45° to the surface of the fluid: to compare the densities of the solid and fluid.

Density of the solid : Density of the fluid :: $32\sqrt{2} - 15 : 32$.

8. Two equal uniform rods, connected by a smooth hinge, and inclined at the same angle to the horizon, are suspended by an elastic vertical string fastened at the hinge, the lower ends of the rods resting in a fluid: to determine the position of equilibrium.

If c be the altitude of the hinge above the surface of the fluid, b the natural length and λ the modulus of elasticity of the string, a the length of each rod, x the length of each rod which is out of the fluid, y the altitude of the hinge above the fluid,

κ the area of a section of either rod, ρ the density of either rod and σ of the fluid, then the position of equilibrium will be determined by the equations

$$x = a\left(1 - \frac{\rho}{\sigma}\right)^{\frac{1}{3}},$$

$$y = c - b - 2b\lambda g\kappa a\left\{\rho - \sigma + \sigma\left(1 - \frac{\rho}{\sigma}\right)^{\frac{1}{3}}\right\}.$$

Section II.

Incompressible Heterogeneous Fluids.

1. Two equal cylinders balance at the extremities of equal arms of a straight lever, when immersed in two fluids the densities of which vary as the depth; the end of one of the cylinders coincides with the surface of the fluid, and the depth of the upper end of the other is equal to n times its altitude: to compare the densities of the two fluids at equal depths.

Let l denote the length of each cylinder, A the area of the base of each; ρ, ρ', the densities of the fluids at the same depth c. Then, since the masses of the two fluids displaced by the cylinders immersed in them must be equal, we have, $\frac{\rho}{c}x$ and $\frac{\rho'}{c}x$, denoting the densities at any depth x,

$$\int_0^l A\frac{\rho}{c}x\,dx = \int_{nl}^{(n+1)l} A\frac{\rho'}{c}x\,dx,$$

$$\int_0^l \rho x\,dx = \int_{nl}^{(n+1)l} \rho' x\,dx,$$

$$\tfrac{1}{2}\rho l^2 = \tfrac{1}{2}\rho'\left\{(n+1)^2 l^2 - n^2 l^2\right\},$$

and therefore $$\rho = (2n+1)\rho',$$

which gives the required ratio between ρ and ρ'.

Instead of equating the masses displaced, it will amount to the same thing if we equate the pressure on the lower end of the one cylinder to the difference of the pressures on the

upper and lower ends of the other cylinder. Thus, p, p', denoting the units of pressure at a depth x in the two fluids,

$$dp = g \cdot \frac{\rho x}{c} \cdot dx, \quad p = \frac{g\rho x^2}{2c},$$

$$dp' = g \frac{\rho' x}{c} dx, \quad p' = \frac{g\rho' x^2}{2c}.$$

The pressure on the base of the one cylinder, putting $x = l$, is equal to

$$Ap = A \cdot \frac{g\rho x^2}{2c} = \frac{Ag\rho l^2}{2c},$$

and the excess of the pressure on the lower end of the other cylinder above that on its upper, $(2n + 1)\, l$ being the value of x at the lower and nl at the upper end, is equal to

$$\frac{A\rho' g\, (n+1)^2 l^2}{2c} - \frac{A\rho' g n^2 l^2}{2c} = \frac{A\rho' g\, (2n+1) l^2}{2c};$$

hence
$$\frac{Ag\rho l^2}{2c} = \frac{Ag\rho'\, (2n+1) l^2}{2c},$$

$$\rho = (2n + 1)\, \rho'.$$

SECTION III.

Bodies floating with constraint under the action of Elastic Fluids.

1. Two equal cylindrical vessels of inconsiderable weights and thicknesses, the one in an upright and the other in an inverted position, are depressed in a fluid to the level of its surface by the exertion of forces P and Q respectively, the surface of the fluid being exposed to the atmospheric pressure: to determine the amount of atmospheric pressure on an area equal to the base of one of the vessels.

Let W denote the required pressure of the atmosphere, ρ the density of the fluid, A the area of the base and h the altitude of either cylinder, and let x denote the length of the portion of the inverted cylinder which is free from water.

Then, for the equilibrium of the upright cylinder,

$$P = g\rho hA \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, for the equilibrium of the inverted one, the air within it, owing to its compression, exerting a force equal to $W\dfrac{h}{x}$ upon the upper end,

$$Q + W = W\frac{h}{x} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, since the pressure of the fluid at the depth x must be equal to that of the air enclosed in the inverted cylinder,

$$W + g\rho xA = W\frac{h}{x} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3),

$$W + P\frac{x}{h} = W\frac{h}{x},$$

and therefore, from (2),

$$Q = P\frac{x}{h} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Multiplying together the equations (2) and (4), we have

$$Q^2 + QW = PW,$$

and therefore $$W = \frac{Q^2}{P - Q}.$$

2. A thin surface, in the form of a paraboloid of revolution, is just immersed in a fluid exposed to the atmosphere, first with its base, next with its vertex downwards, by means of two weights, P, Q: to determine the height of a column of the fluid the weight of which shall be equal to the atmospheric pressure.

If a be the altitude of the paraboloid, and h that of the column of fluid,

$$h = a\left(\frac{P}{Q}\right)^{\frac{1}{2}} \frac{P}{Q - P}.$$

CHAPTER VI.

SPECIFIC GRAVITY.

THE specific gravity of a substance is *measured by*, or, for convenience of language, is said to be *equal* to, the ratio of the weight of a given volume of the substance to that of an equal volume of some assigned standard fluid, generally assumed to be distilled water at a temperature of 60°. The specific gravity of any substance will therefore be represented by the number of units of volume of water which are equal in weight to a unit of volume of the substance in question. A cubic inch of distilled water at a temperature of 60° weighs 253.17 grains. If therefore we know the specific gravity and volume of any substance, we can immediately ascertain its actual weight, by multiplying together the specific gravity of the substance, the number of its units of volume, and the number of units of weight in a unit of volume of water: thus, v denoting the volume of a substance, s its specific gravity, and λ the weight of a unit of volume of water, the weight of the substance will be equal to $sv\lambda$; also, if P denote the weight of a volume of water equal in bulk to the substance, $P = v\lambda$, and therefore the weight of the substance will be equal to Ps.

1. A piece of wood weighs 12lbs, and, when it is annexed to 22lbs of lead, and immersed in water, the whole weighs 8lbs: the specific gravity of lead being 11, to find the specific gravity of the wood.

Let v, v', denote the volumes of the wood and lead respectively; s, s', their respective specific gravities: then, λ denoting the number of pounds in a unit of volume of water,

$$sv\lambda = 12, \quad s'v'\lambda = 22,$$

and
$$sv\lambda + s'v'\lambda - (\lambda v + \lambda v') = 8:$$

hence, $$34 - \frac{12}{s} - \frac{22}{s'} = 8,$$

or, since $s' = 11$, $$34 - \frac{12}{s} - 2 = 8,$$

$$\frac{12}{s} = 24, \quad s = \tfrac{1}{2};$$

a result which shews that the weight of a portion of the wood is equal to half that of an equal bulk of water.

2. A diamond ring weighs $69\frac{1}{2}$ grains, and, when weighed in water, $64\frac{1}{2}$ grains, the specific gravity of diamond being $3\frac{1}{2}$ and that of gold $16\frac{1}{2}$: to find the weight of the diamond.

Let v, v', represent the volumes of the gold and diamond respectively: then, by the conditions of the problem,

$$69\tfrac{1}{2} = \tfrac{33}{2}v\lambda + \tfrac{7}{2}v'\lambda \dots\dots\dots\dots (1),$$

and $$64\tfrac{1}{2} = \tfrac{33}{2}v\lambda + \tfrac{7}{2}v'\lambda - (v + v')\lambda \dots\dots\dots (2).$$

Subtracting the equation (2) from the equation (1), we have

$$5 = \lambda v + \lambda v',$$

and therefore, from (1),

$$69\tfrac{1}{2} = \tfrac{33}{2}(5 - \lambda v') + \tfrac{7}{2}\lambda v',$$

$$139 = 165 - 26\lambda v',$$

$$\lambda v' = 1, \quad \tfrac{7}{2}\lambda v' = 3\tfrac{1}{2},$$

which shews that the weight of the diamond is equal to three grains and a half.

3. A substance A weighs 10 grains in water, and a substance B weighs 14 grains in air, and A and B connected together in water weigh 7 grains: the specific gravity of air being .0013, to determine the specific gravity of B, and to ascertain how many grains of water will be equal to it in bulk.

Let P, P', represent the number of grains in volumes of water equal to the volumes of A, B, respectively, and let s, s', denote the respective specific gravities of these two substances.

Then, by the conditions of the problem,

$$(s - 1)\,P = 10 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(s' - .0013)\,P' = 14 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(s - 1)\,P + (s' - 1)\,P' = 7 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1), (2), (3), we have

$$10 + \frac{14\,(s' - 1)}{s' - .0013} = 7,$$

$$3\,(s' - .0013) = 14\,(1 - s'),$$

$$17s' = 14.0039,$$

and therefore $$s' = .8237.$$

Hence also, from (2),

$$P' = \frac{14}{s' - .0013} = \frac{14}{.8224} = 17.023 \text{ grains}.$$

4. If W_1, W_2, W_3, be the apparent weights of the same substance when immersed in three fluids, the specific gravities of which are s_1, s_2, s_3, respectively, to find the relation between W_1, W_2, W_3, s_1, s_2, s_3.

Let W denote the actual weight of the substance, and P the weight of an equal volume of water; then, by the conditions of the problem,

$$W_1 + s_1 P - W = 0,$$

$$W_2 + s_2 P - W = 0,$$

$$W_3 + s_3 P - W = 0.$$

Eliminating P and W between these three equations by cross multiplication, we get

$$W_1\,(s_3 - s_2) + W_2\,(s_1 - s_3) + W_3\,(s_2 - s_1) = 0.$$

5. When 73 parts by weight of sulphuric acid, the specific gravity of which is equal to 1.8485, are mixed with 27 parts of water, the resulting dilute acid is of a specific gravity equal to 1.6321: to find the amount of the condensation which takes place during the mixture.

Let P denote the number of parts of weight in a quantity of water equal in volume to that of the sulphuric acid, and Q in a quantity of water equal in volume to that of the mixture.

Then, by the conditions of the problem,

$$1.8485 \times P = 73,$$

whence, to four places of decimals, approximately,

$$P = 39.4915 \ldots\ldots\ldots\ldots\ldots (1):$$

and $$1.6321 \times Q = 73 + 27 = 100,$$

whence to four places of decimals,

$$Q = 61.2707 \ldots\ldots\ldots\ldots\ldots (2).$$

Now the condensation of the mixture will be measured by the ratio of the diminution of the volume of the water and acid when mixed together to their united volume before mixture.

Hence $$\text{condensation} = \frac{P + 27 - Q}{P + 27}:$$

but, from (1) and (2),

$$P + 27 = 66.4915, \quad P + 27 - Q = 5.2208;$$

hence, as far as four places of decimals,

$$\text{condensation} = \frac{5.2208}{66.4915} = .0785.$$

6. Two fluids, the volumes of which are v, v', and specific gravities s, s', on being mixed, contract $\left(\frac{1}{n}\right)^{\text{th}}$ part of the sum of their volumes by mutual penetration: to find the specific gravity of the mixture.

Let s_1 denote the specific gravity of the mixture: then, the sum of the weights, before mixture, being

$$(vs + v's')\,\lambda,$$

and, after mixture, being

$$\left(1 - \frac{1}{n}\right)(v + v')\, s_1 \lambda,$$

we have, equating these expressions,

$$\frac{n-1}{n}(v + v')\, s_1 = vs + v's',$$

$$s_1 = \frac{n}{n-1} \cdot \frac{vs + v's'}{v + v'}.$$

7. A crown consists of gold and silver. The crown is totally immersed in a vessel just full of fluid, and the volume of the fluid thus made to flow over is ascertained. A mass of pure gold and a mass of pure silver, the weight of each being equal to that of the crown, are also totally immersed, each of them separately, in the vessel when just filled with fluid, the volume of the overflowing fluid being in each case observed. To determine the volumes of the gold and silver contained in the crown.

Let W denote the weight of the crown, the weight of the pure gold, or the weight of the pure silver; s the specific gravity of gold, s' that of silver; a, a', the volumes of the masses of pure gold and pure silver respectively; β, β', the respective volumes of the gold and silver in the crown, and c the volume of the crown. The quantities a, a', c, are known by the experiment, since the volume of the fluid which escapes from a full vessel is always equal to the volume of the mass totally immersed.

Then, λ denoting the weight of a unit of volume of distilled water,

$$W = \lambda s a, \quad W = \lambda s' a', \quad W = \lambda s \beta + \lambda s' \beta',$$

$$c = \beta + \beta'.$$

From the first three of these equations

$$1 = \frac{\beta}{a} + \frac{\beta'}{a'},$$

and therefore, by the aid of the fourth,

$$1 = \frac{\beta}{a} + \frac{c - \beta}{a'},$$

$$\beta = c \frac{\frac{1}{c} - \frac{1}{a'}}{\frac{1}{a} - \frac{1}{a'}},$$

and, similarly,

$$\beta' = c \frac{\frac{1}{c} - \frac{1}{a}}{\frac{1}{a'} - \frac{1}{a}},$$

"Archimedis verò cum multa miranda inventa et varia fuerint, ex omnibus etiam infinita solertia id quod exponam videtur esse expressum nimiùm. Hiero enim Syracusis auctus regia potestate, rebus bene gestis, cùm auream coronam votivam diis immortalibus in quodam fano constituisset ponendam, manupretio locavit faciendam, et aurum ad sacoma appendit redemptori. Is ad tempus opus manufactum subtiliter regi approbavit, et ad sacoma pondus coronæ visus est præstitisse. Posteaquam indicium est factum, dempto auro tantumdem argenti in id coronarium opus admixtum esse, indignatus Hiero se contemptum, neque inveniens qua ratione id furtum deprehenderet, rogavit Archimedem, uti in se sumeret sibi de eo cogitationem. Tunc is, cum haberet ejus rei curam, casu venit in balneum, ibique cùm in solium descenderet, animadvertit, quantum corporis sui in eo insideret, tantum aquæ extra solium effluere. Itaque cum ejus rei rationem explicationis offendisset, non est moratus, sed exilivit gaudio motus de solio, et nudus vadens domum versus, significabat clara voce invenisse quod quæreret. Nam currens identidem Græcè clamabat *εὕρηκα, εὕρηκα.* Tum vero ex eo inventionis ingressu duas dicitur fecisse massas æquo pondere, quo etiam fuerat corona, unam ex auro, alteram ex argento. Cum ita fecisset, vas amplum ad summa labia implevit aqua, in quo demisit argenteam massam: cujus quanta magnitudo in vase depressa est, tantum aquæ effluxit. Ita exempta massa, quanto minus factum fuerat, refudit, sextario mensus, ut eodem modo quo prius fuerat ad labia æquaretur. Ita ex eo invenit, quantum pondus argenti ad certam aquæ mensuram responderet. Cum id expertus esset, tum auream massam similiter pleno vase dimisit, et ea exempta, eadem ratione mensura addita, invenit ex aqua non tantum defluxisse, sed tantum minùs, quantum minùs magno corpore eodem pondere auri massa esset qùam argenti. Postea vero repleto vase in eadem aqua ipsa corona demissa, invenit plus aquæ defluxisse in coronam, qùam in auream eodem pondere massam, et ita ex eo quod plus defluxerat aquæ in corona, qùam in massa, ratiocinatus, deprehendit argenti in auro mixtionem, et manifestum furtum redemptoris."

Vitruvius: *de Archit.*, lib IX. cap. 3.

8. A body, the weight of which in vacuum is 73.29 grains, loses 24.43 grains by immersion in water: to determine its specific gravity.

Specific gravity = 3.

9. Given the ratio of the magnitudes of two bodies, and of their specific gravities, and the weight of one of them: to find the weight of the other.

Let W, W', denote the respective weights of two bodies A, B; let r denote the ratio of the specific gravity and a of the volume of B to the specific gravity and volume of A. Then

$$W' = arW.$$

Stevin: *Œuvres Mathématiques*, par Albert Girard, p. 497.

10. To find the specific gravity of a substance which weighs 65 grains in vacuum and 44 grains in water.

Its specific gravity is equal to $\frac{65}{21}$.

11. The ratio of the specific gravity of a fluid to that of a solid is 8 : 1; to determine how far a cube, an edge of which is 10 inches, and which is formed out of the solid, will sink in the fluid.

It will sink to the depth of an inch and a quarter.

12. A uniform rod, the length of which is 5 feet 3 inches, and specific gravity .63, floats vertically in water: to determine the length of that portion of the rod which is immersed in the water.

The required length is equal to 3.3075 feet.

13. An hydrometer sinks to a certain depth in a fluid, the specific gravity of which is equal to .82; and, if 60 grains be placed upon it, to the same depth in water: to find the weight of the hydrometer.

Weight of hydrometer = 240 grains.

14. A piece of elm weighs 15 lbs. in vacuum, and a piece of copper, weighing 18 lbs. in vacuum and 16 lbs. in water, is affixed to it, when the compound is found to weigh 6 lbs. in water: to find the specific gravities of the elm and the copper.

If s, s', be the specific gravities of the elm and the copper respectively,

$$s = \tfrac{3}{5}, \quad s' = 9.$$

15. A spherical shell of matter floats in a fluid: having given the ratio of the specific gravities of the fluid and of the shell, the depth of immersion and the radius of the outward surface, to determine the radius of the interior surface of the shell.

If a denote the radius of the exterior and a' of the interior surface of the shell, n the ratio of the specific gravity of the fluid to that of the shell, and c the depth of the lowest point of the outer surface of the shell below the surface of the fluid, then

$$a' = a\left\{1 - \frac{3}{4}\frac{nc^2}{a^2} + \frac{1}{4}n\frac{c^3}{a^3}\right\}^{\frac{1}{3}}.$$

Leibnitz: *Miscellanea Berolinensia*, tom. I. p. 127.
Herman: *Phoronomia*, p. 157.

16. A bottle provided with a glass stopper weighs 6 ounces when empty; when it is immersed in water, its apparent weight is half an ounce; but when the stopper is loosened and the water let in, its apparent weight is 4 ounces: to find the specific gravity of the glass, and to determine in cubic inches the interior capacity of the bottle, supposing a cubic foot of water to weigh 1000 ounces.

The specific gravity of the glass is equal to 3, and the volume of the content of the bottle in cubic inches is 6.048.

17. Two quarts of a liquid, the specific gravity of which is 1.35, are mixed with three quarts of another, and the specific gravity of the mixture is 1.5: to find that of the second liquid.

The required specific gravity = 1.6.

18. The specific gravities of cork and lead being respectively .24 and 11.325, to find the volumes of cork and lead in a mass of 10 cubic feet, that it may float in distilled water, one half being immersed.

If a, a', denote the numbers of cubic feet of cork and lead respectively, in the mass,

$$a = \frac{21650}{2217}, \quad a' = \frac{520}{2217}.$$

19. A solid cylinder floats with its axis vertical in two fluids which do not mix, the two surfaces of the upper fluid dividing the cylinder into three equal parts: the specific gravities of the fluids being .579 and .972, to determine the specific gravity of the cylinder.

The specific gravity of the cylinder = .517.

20. The tables of specific gravity give

Mercury, 13.6; Iron, 7.8; Water, 1:

to find how deep a uniform rod of iron must be immersed, when floating vertically in a vessel of mercury, the length of the rod being $6\frac{1}{4}$ inches.

The length of the rod which is immersed in the mercury must be nearly equal to $3\frac{3}{5}$ inches.

21. If water be resting upon a fluid the specific gravity of which is $\frac{3}{2}$, and a body float with two parts in this fluid and one part in the water; to find the specific gravity of the body.

The specific gravity of the body = $\frac{4}{3}$.

22. A thin rod AB (fig. 19) is suspended by a string at A, its lower end B resting in a fluid, the surface of which is intersected by the rod in the point C; having given the length of the rod, and the ratio of its specific gravity to that of the fluid, to determine the portion AC of the rod which is out of the fluid.

If $AB = a$, $AC = x$, and n = the ratio of the specific gravity of the rod to that of the fluid,

$$x^2 = (1 - n)\,a^2.$$

Herman: *Phoronomia*, p. 159.

CHAPTER VII.

STABILITY OF EQUILIBRIUM.

If a body floating in a fluid, when displaced from its position of rest, have a tendency to return to its original position, the equilibrium is said to be *stable*, and, if to recede still further from its original position, the equilibrium is said to be *unstable in regard to the displacement.* We shall confine our attention to the general case of a body which is symmetrical with regard to a vertical section perpendicular to its plane of floatation, and shall suppose the displacement to be of such a nature that all the particles of the body are made to move parallel to the section of symmetry.

Section I.

Finite Displacements.

In this section we shall suppose the displacement of the body from its position of rest to be finite. Conceive the body to be placed in a position not coinciding with that of rest. Let G (fig. 20) be the centre of gravity of the body, H that point of the body which, when the body is in its position of rest, coincides with that of the fluid displaced, and H' the centre of gravity of the fluid displaced by the body in its actual position. The three points G, H, H', will evidently be in the section of symmetry. Let M be the point in which the vertical through H' meets the indefinite line through G and H. The body will be acted on by two forces; a force, equal to the weight of the fluid actually displaced, acting in the direction $H'M$, and the weight of the body acting downwards through G.

If the point M be above G, the action of the fluid will therefore tend to twist the line HG into a vertical position, or to restore the body to its position of rest, and if the point M be below G, to cause the inclination of HG to the vertical to increase, or to twist the body still more from its position of rest. If therefore M be above G, the equilibrium will be ***stable***, and if below G, it will be ***unstable***. If M coincide with G, the equilibrium will be neither ***stable*** nor ***unstable***, but ***indifferent*** or *neutral*.

The discovery of the circumstances necessary for the stability of floating bodies is due to the genius of Archimedes, by whom the conditions for the stability of floating segments of spheres and paraboloids of revolution were fully discussed in his treatise entitled, in the edition of Tartalea's Translation by Commandine, *De iis quæ vehuntur in aquâ.*

1. To ascertain whether a segment of a sphere, floating with its vertex downwards, is in a position of stable or of unstable equilibrium.

Let AOB (fig. 21) be the segment of the sphere, displaced from its position of rest, AB being its base, and O its vertex. Let C be the centre of the sphere of which the segment is a portion. Join CO: then the centre of gravity G of the segment will be somewhere in the line CO. Let EF be the section of the segment made by a plane coincident with the surface of the water, and let V be the vertex of the segment EVF: join CV. Then the action of the fluid upon the segment AOB will be equivalent to a force, equal to the weight of the fluid displaced by the segment EVF, acting upwards in the line VC. Hence the weight of the solid, acting downwards through G, and the action of the fluid, will tend to twist the axis CO of the segment AOB into a vertical position. The equilibrium of the segment will therefore be stable.

This proposition is due to Archimedes. "Si aliqua magnitudo solida levior humido, quæ figuram portionis sphæræ habeat, in humidum demittatur, ita ut basis portionis non tangat humidum: figura insidebit recta, ita ut axis portionis sit secundum perpendicularem. Et si ab aliquo inclinetur figura, ut

basis portionis humidum non contingat, non manebit inclinata si demittatur, sed recta restituetur."

De iis quæ vehuntur in aquâ, lib. I. Prop. 8.

2. A paraboloid of revolution is immersed in a fluid, with its axis inclined at a certain angle to the horizon, so that the circular base of the paraboloid is entirely out of the fluid; the weight of the volume of the fluid displaced being equal to that of the paraboloid, to determine whether the action of the fluid will tend to twist the axis of the paraboloid towards the vertical or in the contrary direction.

Let ABC (fig. 22) be the paraboloid, A its vertex; QQ' the axis major of the elliptic section of the solid made by the surface of the fluid; S the focus, PT a tangent to the paraboloid in the plane QAQ' parallel to QQ'; G the centre of gravity of the solid, which will lie in the axis AO, H' of the fluid displaced, M the intersection of the axis of the paraboloid with the vertical line through H': let V be the middle point of the line QQ'; join VP: it is easily seen that the point H' will lie in VP. Let VW be a line drawn at right angles to the plane QAQ' to meet the surface of the paraboloid in the point W. Let vq, vw, be lines, in a section of the solid parallel to QWQ', analogous to VQ, VW; and let $Pv = x'$. Let $4m$ be the latus rectum of the parabola, $PV = x$.

Then, θ being the inclination of the axis of the paraboloid to the horizon,

$$qv^2 = 4SP.x' = \frac{4m}{\sin^2\theta}.x',$$

and, a plane section PwW of the paraboloid at right angles to the plane QAQ', being a parabola similar to the generating parabola,

$$wv^2 = 4mx':$$

hence

$$qv.wv = \frac{4m}{\sin\theta}.x',$$

and therefore the area of the elliptic section through vq, parallel to QWQ', qv, vw, being its semi-axes, will be equal to

$$\pi.qv.wv = \frac{4\pi mx'}{\sin\theta}.$$

Hence, putting $PH' = \bar{x}$,

$$\bar{x}.\int_0^x \frac{4\pi m x'}{\sin\theta}.\sin\theta\, dx' = \int_0^x \frac{4\pi m x'}{\sin\theta}.x'.\sin\theta\, dx',$$

$$\bar{x}.\int_0^x x' dx' = \int_0^x x'^2 dx',$$

$$\bar{x}.\tfrac{1}{2}x^2 = \tfrac{1}{3}x^3,$$

$$\bar{x} = \tfrac{2}{3}x.$$

Again,

$$TM.\cos\theta = PT + PH'.\cos\theta = 2SP\cos\theta + PH'.\cos\theta,$$

and therefore $\quad TM = 2SP + \frac{2}{3}x = \dfrac{2m}{\sin^2\theta} + \frac{2}{3}x.$

Also $\quad TA = ST - m = \dfrac{m}{\sin^2\theta} - m:$

hence $\quad AM = m + \dfrac{m}{\sin^2\theta} + \frac{2}{3}x \ldots\ldots\ldots\ldots\ldots(1).$

Now, the weight of the volume of the fluid displaced being equal to that of the solid, we have, putting $AO = a$, $\rho =$ the density of the solid, $\sigma =$ that of the fluid,

$$2\pi m a^2.\rho = \sigma\int_0^x \frac{4\pi m x'}{\sin\theta}.\sin\theta\, dx' = 2\pi m\sigma x^2,$$

whence $\quad x = \left(\dfrac{\rho}{\sigma}\right)^{\frac{1}{2}}.a.$

We have then $\quad AM = m + \dfrac{m}{\sin^2\theta} + \dfrac{2}{3}\left(\dfrac{\rho}{\sigma}\right)^{\frac{1}{2}}a.$

In order that the fluid may tend to twist AO into a vertical position, we must have

$$AM > AG,$$

or $$m + \frac{m}{\sin^2\theta} + \frac{2}{3}\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}}a > \tfrac{2}{3}a,$$

or $$\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} > \frac{a - \frac{3}{2}\left(m + \dfrac{m}{\sin^2\theta}\right)}{a},$$

$$\frac{\rho}{\sigma} > \frac{\left\{a - \frac{3}{2}\left(m + \dfrac{m}{\sin^2\theta}\right)\right\}^2}{a^2} \ldots\ldots\ldots\ldots(2).$$

If $\frac{\rho}{\sigma}$ be less than this expression, the axis of the paraboloid will recede still further from the vertical. The condition (2) will therefore ensure the stability of the paraboloid floating with its axis vertical. This condition will be satisfied *à fortiori* if

$$\frac{\rho}{\sigma} > \frac{(a - \frac{3}{2} \cdot 2m)^2}{a^2}.$$

This result agrees with that given by Archimedes, *De iis quæ vehuntur in aquâ*, lib. II. Prop. 4. "Recta portio conoidis rectanguli, quando fuerit humido levior, et axem habuerit majorem, quàm sesquialterum ejus, quæ usque ad axem: si in gravitate ad humidum æqualis molis non minorem proportionem habeat ea, quam quadratum, quod sit ab excessu, quo axis major est, quàm sesquialter ejus, quæ usque ad axem, habet ad quadratum, quod ab axe; demissa in humidum, ita ut basis ipsius humidum non contingat, et posita inclinata, non manebit inclinata, sed recta restituetur."

COR. 1. Suppose the point C to be in contact with the surface of the fluid. Then

$$(4ma)^{\frac{1}{2}} = QV. \sin\theta + \frac{2m}{\tan\theta}$$

$$= (4mx)^{\frac{1}{2}} + \frac{2m}{\tan\theta},$$

or, since x is equal to $\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} a$,

$$a^{\frac{1}{2}}\left\{1 - \left(\frac{\rho}{\sigma}\right)^{\frac{1}{4}}\right\} = \frac{m^{\frac{1}{2}}}{\tan\theta}.$$

Hence, from (2), that the axis of the paraboloid may tend to become vertical, we have the condition

$$\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} > 1 - \frac{3m}{a} - \frac{3}{2}\left\{1 - 2\left(\frac{\rho}{\sigma}\right)^{\frac{1}{4}} + \left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}}\right\},$$

$$\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} - \frac{6}{5}\left(\frac{\rho}{\sigma}\right)^{\frac{1}{4}} > -\frac{1}{5} - \frac{6m}{5a},$$

$$\left\{\left(\frac{\rho}{\sigma}\right)^{\frac{1}{4}} - \frac{3}{5}\right\}^2 > \frac{4a - 30m}{25a},$$

a condition which will be satisfied provided that

$$a < \tfrac{15}{4} . 2m,$$

whatever be the value of $\frac{\rho}{\sigma}$.

This result comprehends Prop. 6., lib. II., of the Treatise of Archimedes:—"Recta portio conoidis rectanguli, quando levior humido axem habuerit majorem quidem quàm sesquialterum ejus, quæ usque ad axem, minorem vero, quàm ut ad eam, quæ usque ad axem, proportionem habeat, quam quindecim ad quatuor; in humidum demissa adeo, ut basis ipsius contingat humidum, nunquam consistet inclinata ita, ut basis in uno puncto humidum contingat."

Cor. 2. In order that the axis of the paraboloid may tend to assume a vertical position, the sufficient and necessary condition consists in the inequality

$$AM > AG;$$

hence by (1), since AG is equal to $\frac{2}{3}a$, we have

$$m + \frac{m}{\sin^2\theta} + \tfrac{2}{3}x > \tfrac{2}{3}a,$$

or

$$a < \tfrac{3}{2}m + \tfrac{3}{2}\frac{m}{\sin^2\theta} + x$$

$$< \tfrac{3}{2} . 2m + \tfrac{3}{2}m \cot^2\theta + x,$$

and therefore, *à fortiori*, the axis will tend to become vertical, whatever be the ratio between the density of the fluid and paraboloid, and whatever volume of the paraboloid be immersed, the base being supposed to be entirely out of the fluid, provided that a be less than $\frac{3}{2} . 2m$. This conclusion agrees with Prop. 2., lib. II., of the Treatise of Archimedes:—"Recta portio conoidis rectanguli, quando axem habuerit minorem, quam sesquialterum ejus, quæ usque ad axem, quamcunque proportionem habens ad humidum in gravitate; demissa in humidum, ita ut basis ipsius humidum non contingat; et posita inclinata, non manebit inclinata; sed recta restituetur. Rectam dico consistere talem portionem, quando planum quod ipsam secuit, superficiei humidi fuerit æquidistans."

COR. 3. If the axis tend to move neither one way nor the other, that is, if the paraboloid be in a position of rest with its axis inclined at an angle θ to the horizon, we must have

$$AM = AG,$$

and therefore $$m + \frac{m}{\sin^2\theta} + \frac{2}{3}\left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} a = \tfrac{2}{3}a,$$

or $$\sin^2\theta = \frac{\frac{3}{2}m}{a - \left(\frac{\rho}{\sigma}\right)^{\frac{1}{2}} a - \frac{3}{2}m}.$$

These conclusions are established by Archimedes, lib. II. Prop. 8.

3. A paraboloid of revolution is immersed in a fluid, with its axis inclined at a certain angle to the horizon, so that the circular base of the paraboloid is entirely within the fluid: the weight of the volume of the fluid displaced being equal to that of the paraboloid, to determine whether the action of the fluid will tend to twist the axis of the paraboloid towards the vertical or in the contrary direction.

If a denote the length of the axis of the paraboloid, $4m$ its latus rectum, θ the inclination of its axis to the horizon, ρ the density of the paraboloid, and σ that of the fluid, the axis will tend to approach or recede from the vertical accordingly as $\frac{\rho}{\sigma}$ is less or greater than the expression

$$1 - \frac{\left\{a - \frac{3}{2}\left(m + \frac{m}{\sin^2\theta}\right)\right\}^2}{a^2},$$

or, if θ be nearly $\frac{1}{2}\pi$, than the expression $\dfrac{a^2 - (a - \frac{3}{2}.2m)^2}{a^2}$.

Archimedes; *De iis quæ vehuntur in aquâ*, lib. II. Prop. 5. "Recta portio conoidis rectanguli, quando levior humido axem habuerit majorem, quàm sesquialterum ejus, quæ usque ad axem; si ad humidum in gravitate non majorem proportionem habeat, quàm excessus, quo quadratum quod fit ab axe majus est quadrato, quod ab excessu, quo axis major est, quàm sesquialter ejus, quæ usque ad axem, ad quadratum, quod ab axe: demissa in humidum, ita ut basis ipsius tota sit in humido;

et posita inclinata non manebit inclinata, sed restituetur ita, ut axis ipsius secundum perpendicularem fiat." From the investigations which present themselves in the solution of this problem it is easy to ascertain that, if a be less than $\frac{3}{4} \cdot 2m$, the axis will tend to assume a vertical position whatever be the ratio between the density of the paraboloid and that of the fluid. Archimedes, prop. 3. lib. II: "Recta portio conoidis rectanguli, quando axem habuerit minorem, quam sesquialterum ejus, quæ usque ad axem, quamcunque proportionem habens ad humidum in gravitate; demissa in humidum, ita ut basis ipsius tota sit in humido; et posita inclinata, non manebit inclinata, sed ita restituetur, ut axis ipsius secundum perpendicularem fiat."

Section II.

Indefinitely small Displacements. Metacentre.

Suppose that, adopting the construction and notation of the preceding section, the angular displacement, supposed to be impressed upon the solid about a certain line in the plane of floatation at right-angles to the section of symmetry, is indefinitely small, and that the volume of the displaced fluid is the same before and after the displacement, that is, such that its weight is equal to that of the floating body; this line of rotation, as is proved in all systematic treatises on Hydrostatics, will pass through the centre of gravity of the plane of floatation. The ultimate position of M is called the *metacentre* of the floating body. The point H' will ultimately coincide with H, and the magnitude of HM will be defined by the formula

$$HM = \frac{Ak^2}{V},$$

A being the area of the plane of floatation, Ak^2 its moment of inertia about its axis of rotation, and V the volume of the fluid displaced. For small displacements the stability, neutrality, instability of the equilibrium will correspond respectively to the conditions

$$\frac{Ak^2}{V} > HG, \quad \frac{Ak^2}{V} = HG, \quad \frac{Ak^2}{V} < HG;$$

and putting $HG = \gamma$, the quantity $\frac{Ak^2}{V} - \gamma$ may be taken as a measure of the stability.

The term *metacentre* was first used by Bouguer, in a work entitled *Traité du Navire*, published in 1746, and the formula which we have given for computing the magnitude of HM, is substantially the same as that given by Bouguer in his treatise, liv. II. sect. ii. chap. 3, p. 262.

1. To find the position of the metacentre of a rectangular parallelopiped, supposed to receive an angular displacement about a line in its plane of floatation at right angles to two of its faces.

Let a = the length, and b = the breadth of its plane of floatation, and c = the depth of the parallelopiped, the line of rotation being supposed to be perpendicular to the length a.

Then $$Ak^2 = \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} bx^2 dx = \tfrac{1}{12} ba^3:$$

also, ρ being the density of the solid, σ of the fluid, and c' the depth to which the parallelopiped is immersed,

$$\sigma c' = \rho c,$$

which determines c'.

Hence $$HM = \frac{\frac{1}{12} a^3 b}{abc'} = \frac{a^2}{12c'}.$$

This result shews that, to ensure the stability of the body, the greatest height of its centre of gravity above the centre of gravity of the displaced fluid must not be greater than $\frac{a^2}{12c'}$. Supposing Noah's ark to have been, like the Chinese junks, in the form of a parallelopiped, Bouguer (*Traité du Navire*, p. 256) observes, "S'il s'agissoit en particulier de l'Arche de Noé, dont la largeur étoit de 50 coudées, et qu' on supposât que ce bâtiment enfonçoit dans les eaux du déluge de 10 coudées, on trouvera que le métacentre M étoit élevé de $20\frac{5}{6}$ coudées au-dessus du centre de gravité de la carene, et par conséquent de $15\frac{5}{6}$ au-dessus de la surface de la mer, et de $25\frac{5}{6}$ au-dessus du fond de la cale. Il étoit difficile, ou plutôt il n'étoit pas possible que le centre de gravité se trouvât porté

à une si grande élévation, puisque toute l'Arche n'avoit que 30 coudées de hauteur. Ainsi l'inclinaison de ce bâtiment ne pouvoit jamais devenir trop grande; il n'y avoit rien à craindre de ce côté pour les précieux restes du genre humain."

2. To find the metacentre of a right cone floating in a fluid.

Let R (fig. 23) be the radius OA of its base, H its altitude OV; ρ the density of the cone and σ that of the fluid. Then, M being the metacentre, and H the centre of gravity of the fluid displaced aVb, we have

$$HM = \frac{Ak^2}{V},$$

V being the volume of aVb, and Ak^2 the moment of inertia of the circular area ab about a diameter.

For equilibrium we must have, h being the length Vo of the immersed portion of the axis, and r the radius ao of the circular area ab,

$$\tfrac{1}{3}\pi R^2 . H . g\rho = \tfrac{1}{3}\pi r^2 h . g\sigma,$$

or

$$R^2 . H . \rho = r^2 h\sigma;$$

whence, observing that $h : r :: H : R$,

$$\left.\begin{aligned} H^3\rho &= h^3\sigma \\ \text{and} \qquad R^3\rho &= r^3\sigma \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

Again, $$V = \tfrac{1}{3}\pi r^2 h \dots\dots\dots\dots\dots\dots (2),$$

and $$Ak^2 = \tfrac{1}{4}\pi r^4 \dots\dots\dots\dots\dots\dots (3).$$

From (2) and (3) we have for HM the value

$$\frac{\tfrac{1}{4}\pi r^4}{\tfrac{1}{3}\pi r^2 h} = \frac{3r^2}{4h};$$

and therefore, from (1),

$$HM = \frac{3}{4}\,\frac{R^2\left(\dfrac{\rho}{\sigma}\right)^{\frac{2}{3}}}{H\left(\dfrac{\rho}{\sigma}\right)^{\frac{1}{3}}} = \frac{3R^2\rho^{\frac{1}{3}}}{4H\sigma^{\frac{1}{3}}}.$$

COR. Since $$VG = \tfrac{3}{4}H,$$

and $$VH = \tfrac{3}{4}h = \tfrac{3}{4}H\left(\frac{\rho}{\sigma}\right)^{\frac{1}{3}};$$

we have $$HG = \tfrac{3}{4}H\left\{1 - \left(\frac{\rho}{\sigma}\right)^{\frac{1}{3}}\right\}.$$

In order, therefore, that the equilibrium may be stable, we must have

$$\frac{3R^2\rho^{\frac{1}{3}}}{4H\sigma^{\frac{1}{3}}} > \tfrac{3}{4}H\left\{1 - \left(\frac{\rho}{\sigma}\right)^{\frac{1}{3}}\right\},$$

$$\frac{R^2}{H^2} > \frac{\sigma^{\frac{1}{3}} - \rho^{\frac{1}{3}}}{\rho^{\frac{1}{3}}}.$$

If the equilibrium be neutral,

$$\frac{R^2}{H^2} = \frac{\sigma^{\frac{1}{3}} - \rho^{\frac{1}{3}}}{\rho^{\frac{1}{3}}}.$$

Daniel Bernoulli: *Comment. Acad. Petrop.* 1738, p. 163.

3. A cone, the vertical angle of which is $\frac{1}{2}\pi$, and of which the density throughout any section perpendicular to the axis varies inversely as the square of the distance of the section from the vertex, floats in a fluid, the density of which is m times the least density of the cone: to determine the value of m when the equilibrium is one of indifference.

Adopting the figure and notation of the preceding problem, we have, observing that in the present case r is equal to h,

$$HM = \frac{3r^2}{4h} = \tfrac{3}{4}h,$$

and therefore $\qquad VM = VH + HM = \tfrac{3}{2}h.$

Again, $\frac{\mu}{H^2}$ denoting the density of the base of the cone, and r_1 the radius of a circular section at a distance h_1 from the vertex, the mass of the cone will be equal to

$$\int_0^H \pi r_1^2 \,.\, dh_1 \,.\, \frac{\mu}{h_1^2} = \int_0^H \pi h_1^2 \,.\, dh_1 \,.\, \frac{\mu}{h_1^2}$$

$$= \pi\mu \int_0^H dh_1 = \pi\mu H.$$

Also, the density of the fluid being equal to $\frac{m\mu}{H^2}$, the mass of the fluid displaced will be equal to

$$\tfrac{1}{3}\pi r^2 h \,.\, \frac{m\mu}{H^2} = \tfrac{1}{3}\pi\mu m \,\frac{h^3}{H^2}.$$

Equating the mass of the cone to that of the fluid displaced, we get

$$\pi\mu H = \tfrac{1}{3}\pi\mu m \frac{h^3}{H^2},$$

$$H^3 = \tfrac{1}{3}mh^3,$$

$$VG = \tfrac{3}{4}H = \tfrac{3}{4}h\left(\frac{m}{3}\right)^{\frac{1}{3}}.$$

Since the equilibrium is to be one of indifference, we must have

$$VG = VM,$$

and therefore

$$\tfrac{3}{2}h = \tfrac{3}{4}h\left(\frac{m}{3}\right)^{\frac{1}{3}}, \quad 2 = \left(\frac{m}{3}\right)^{\frac{1}{3}}, \quad m = 24.$$

4. To find the position of the metacentre of a right-angled triangular board, of which the right angle is immersed in a fluid, the opposite side being horizontal.

The metacentre will be as much above the surface of the fluid as the centre of gravity of the fluid displaced is below it.

Bouguer: *Traité du Navire*, p. 266.

5. A uniform rectangular board floats vertically in a fluid; supposing the vertical length of the board to be to the horizontal in the ratio of $\sqrt{2}$ to $\sqrt{3}$, and the board to be so disturbed as to be still in a vertical plane, to find the ratio of the density of the board to that of the fluid, that the equilibrium may be neutral.

The density of the board must be equal to half that of the fluid.

6. To find the metacentre of a square lamina floating in a fluid with one side horizontal, when the specific gravity of the lamina is equal to two-thirds of that of the fluid.

If a denote the length of a side of the square, and x the altitude of the metacentre above the lowest of the two horizontal sides,

$$x = \tfrac{11}{24}a.$$

7. To determine the condition that a homogeneous cylinder may float with stability in a fluid, its axis being supposed to be vertical.

If r be the radius and h the altitude of the cylinder, and ρ, σ, the respective densities of the cylinder and of the fluid, the sufficient and necessary condition will be expressed by the inequality
$$\frac{r^2}{h^2} > \frac{2\rho(\sigma-\rho)}{\sigma^2}.$$

Daniel Bernoulli: *Comment. Acad. Petrop.* 1738, p. 162.
Encycl. Metrop., Mix. Sc., Vol. I. p. 193.

8. To investigate the nature of the equilibrium of a floating sphere.

The equilibrium will always be neutral.

9. To find the least density of a right-angled cone which can float in stable equilibrium with its vertex downwards in a given fluid.

The density of the cone must not be less than one-eighth of the density of the fluid.

10. A paraboloid, with its axis vertical and vertex downwards, floats in a fluid with half its axis immersed, in a position of indifferent equilibrium: to determine the ratio of the height of the paraboloid to the latus rectum of the generating parabola.

If h be the height and l the latus rectum,
$$\frac{h}{l} = \frac{3}{2}.$$

11. A rectangular board floats vertically with two of its sides parallel to the horizon: to find the measure of its stability.

If ρ denote the density of the board, σ that of the fluid, a the breadth and b the altitude of the board, then MG, the magnitude of which may be regarded as a measure of the stability, will be equal to
$$\frac{a^2\sigma}{12b\rho} - \tfrac{1}{2}b\left(1 - \frac{\rho}{\sigma}\right).$$

If this quantity be negative, the equilibrium will be unstable, and if zero, it will be neutral.

Encycl. Metrop., Mix. Sc., Vol. I. p. 193.

12. A prism, the section of which perpendicular to its axis is an isosceles triangle, the vertical angle of which is given, floats in water with its vertical angle immersed: to determine the specific gravity of the prism that its equilibrium may be stable.

If α be the vertical angle, and s the specific gravity, the equilibrium will be stable, neutral, or unstable respectively, according to the following conditions,

$$s > \left(\cos\frac{\alpha}{2}\right)^4, \quad s = \left(\cos\frac{\alpha}{2}\right)^4, \quad s < \left(\cos\frac{\alpha}{2}\right)^4.$$

SECTION III.

Stability of Bodies floating under constraint.

1. A uniform homogeneous rod AB (fig. 24), one end A of which is attached to a point at a given depth within a fluid, is placed at a certain inclination to the horizon: to determine whether the rod will tend to assume a vertical position or to recede further from it.

Let $AB = a$, $AH = h$, H being the point in the surface through which the vertical line AH passes; let $AM = x$, AM being the portion of the rod which is immersed in the fluid. Also let $\angle AMH = \theta$, $\rho =$ the density of the rod, and $\sigma =$ that of the fluid.

Then, κ denoting the area of a perpendicular section of the rod, the moment of the fluid displaced about A will be equal to

$$\kappa\sigma x \,.\, \tfrac{1}{2}x \cos\theta = \tfrac{1}{2}\kappa\sigma x^2 \cos\theta;$$

and the moment of the rod about A will be equal to

$$\kappa\rho a \,.\, \tfrac{1}{2}a \cos\theta = \tfrac{1}{2}\kappa\rho a^2 \cos\theta.$$

Hence, that the rod may tend to become vertical, we must have

$$\sigma x^2 \cos\theta > \rho a^2 \cos\theta,$$

or

$$\sigma x^2 > \rho a^2,$$

and therefore, since $x \sin\theta = h$,

$$\sigma h^2 > \rho a^2 \sin^2\theta,$$

$$\sin\theta < \left(\frac{\sigma}{\rho}\right)^{\frac{1}{2}} \frac{h}{a} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

We have supposed the rod to be only partially immersed: supposing it however to be entirely within the fluid, it is plain that, if it have a tendency to assume a vertical position, we must have

$$\tfrac{1}{2}\kappa\sigma a^2 \cos\theta > \tfrac{1}{2}\kappa\rho a^2 \cos\theta,$$

or the sufficient and necessary condition is expressed by the inequality

$$\sigma > \rho.$$

Under the supposition of the partial immersion of the rod, since, if θ be sufficiently small, the inequality (1) will always be satisfied, it is evident that the rod's vertical position of equilibrium will be one of stability at least for small displacements.

2. A prolate spheroid is placed with its axis vertical in a vessel with a horizontal base, into which fluid is poured: to determine the height to which the fluid must rise in order that the equilibrium may be stable for small displacements.

Let AB (fig. 25) be the axis of the spheroid when slightly displaced from its vertical position, so as still to be in contact with the bottom of the vessel.

Let G be the middle point of AB, or the centre of gravity of the spheroid; M the intersection of AB and the vertical line through the centre of gravity of the fluid displaced in the new position of the spheroid; N the intersection of AB and the vertical line in which the reaction of the bottom of the vessel on the spheroid is exerted. Also let H be the point in the line AB, which, in the spheroid's position of rest, coincides with the centre of gravity of the fluid displaced.

Let V denote the volume of the fluid displaced, which will be the same after as before the displacement, since the centre of gravity of the section of the solid, made by the surface of the fluid, will be, for the very small displacement, still in the surface of the fluid. Let V' denote the volume of the spheroid. Let ρ denote the density of the solid, σ that of the fluid; a, b, the semi-axes, major and minor.

Then, the reaction through N being equal to

$$(V'\rho - V\sigma)\,g,$$

it will be sufficient and necessary for the stability of equilibrium that

$$(V'\rho - V\sigma) . GN < \sigma V . MG$$
$$< \sigma V . (HM - AG + AH).$$

Now, when the depth of the fluid is h,

$$V = \pi \int y^2 dx = \pi \frac{b^2}{a^2} \int_0^h (2ax - x^2)\, dx = \pi \frac{b^2}{a^2}(ah^2 - \tfrac{1}{3}h^3),$$

$$V' = \tfrac{4}{3}\pi ab^2,$$

$$GN = a - \frac{b^2}{a},$$

$$V . HM = \tfrac{1}{4}\pi y^4 = \tfrac{1}{4}\pi \frac{b^4}{a^4}(2ah - h^2)^2,$$

$$V . AH = \pi \int_0^h xy^2 dx = \pi \frac{b^2}{a^2} \int_0^h (2ax^2 - x^3)\, dx = \pi \frac{b^2}{a^2}(\tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4).$$

Hence

$$(\tfrac{4}{3}\pi ab^2\rho - V\sigma)\left(a - \frac{b^2}{a}\right) < \sigma \left\{\tfrac{1}{4}\pi \frac{b^4}{a^4}(2ah - h^2)^2 - Va + \pi \frac{b^2}{a^2}(\tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4)\right\},$$

$$\tfrac{4}{3}\pi\rho ab^2\left(a - \frac{b^2}{a}\right) < \sigma \left\{\tfrac{1}{4}\pi \frac{b^4}{a^4}(2ah - h^2)^2 - V\frac{b^2}{a} + \pi \frac{b^2}{a^2}(\tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4)\right\}$$

$$< \sigma \left\{\tfrac{1}{4}\pi \frac{b^4}{a^4}(2ah - h^2)^2 - \pi \frac{b^4}{a^3}(ah^2 - \tfrac{1}{3}h^3) + \pi \frac{b^2}{a^2}(\tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4)\right\},$$

$$\tfrac{4}{3}\rho(a^2 - b^2) < \frac{\sigma}{a^2}\left\{\frac{1}{4}\frac{b^2}{a^2}(2ah - h^2)^2 - \frac{b^2}{a}(ah^2 - \tfrac{1}{3}h^3) + \tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4\right\}$$

$$< \frac{\sigma}{a^2}\left\{\frac{1}{4}\frac{b^2}{a^2}(h^4 - 4ah^3) + \frac{b^2}{3a}h^3 + \tfrac{2}{3}ah^3 - \tfrac{1}{4}h^4\right\}$$

$$< \frac{\sigma}{a^2}\left\{\frac{1}{4}\frac{b^2 - a^2}{a^2}h^4 - \frac{2b^2}{3a}h^3 + \tfrac{2}{3}ah^3\right\}$$

$$< \frac{\sigma}{a^2}\left\{\frac{1}{4}\frac{b^2 - a^2}{a^2}h^4 - \frac{2}{3}\frac{b^2 - a^2}{a}h^3\right\},$$

$$\tfrac{4}{3}\rho < \frac{\sigma}{a^3}\left(\tfrac{2}{3}h^3 - \frac{1}{4}\frac{h^4}{a}\right),$$

$$2\rho < \sigma\frac{h^3}{a^3}\left(1 - \frac{3h}{8a}\right),$$

or, for the smallest value of h,

$$2\rho = \sigma\frac{h^3}{a^3}\left(1 - \frac{3h}{8a}\right).$$

3. A homogeneous right cone, the vertex of which is attached to a fixed point at a certain depth within a fluid, is placed with its axis vertical, its base being supposed to be out of the fluid: to determine the condition that the equilibrium of the cone may be stable for indefinitely small displacements.

If ρ denote the density of the cone, σ that of the fluid, $2a$ the vertical angle of the cone, h its altitude, and h' the depth of the vertex below the surface, the sufficient and necessary condition for stability is expressed by the inequality

$$\frac{h'^4}{h^4} > \frac{\rho \cos^2 a}{\sigma}.$$

CHAPTER VIII.

EQUILIBRIUM OF VESSELS CONTAINING FLUID.

SECTION I.

Equilibrium.

THE system consisting of a vessel and the contained fluid, will be subject to the action of three forces, viz. the weight of the vessel, the weight of the fluid, and the reaction of the surface upon which the vessel is supported. The conditions of the equilibrium may therefore be ascertained as in ordinary statical problems.

1. A cubical vessel, half filled with fluid, rests upon a horizontal plane; to find the greatest angle through which the plane can be turned before the vessel overturns, the lowest edge of the cube remaining horizontal, and all sliding being prevented.

It is evident that the angle may be as great as 45°, for, under this condition, the centre of gravity of the fluid contained in it, and of the vessel itself, will lie in the vertical line through the middle point of its lowest edge. It may easily be ascertained that this is the greatest elevation. For, supposing the elevation to be greater, in which case a portion of the fluid will have escaped, let $ABCD$ (fig. 26) be a vertical section of the vessel through the centre of gravity G of the fluid, at right angles to the inclined plane. Let AK be the section of the surface of the fluid, which will be a horizontal line. Join A, G, and produce the line AG to M, which will be the middle point of BK. Draw GH at right angles to BK: join GB.

Then $\tan \angle GBH = \frac{GH}{BH} = \frac{\frac{1}{3}AB}{\frac{2}{3}BM} = \frac{\frac{1}{3}AB}{\frac{1}{3}BK} = \frac{AB}{BK}$.

But, if θ be the inclination of the inclined plane to the horizon, the tangent of the acute angle between the vertical through B and the inclined plane will be equal to cot θ. Hence, supposing the weight of the vessel itself to be inconsiderable, we must have, that equilibrium may be preserved,

$$\tan \angle GBH \angle \text{ or } = \cot \theta \ldots\ldots\ldots\ldots (1);$$

a condition necessary *à fortiori*, if the weight of the vessel be not inconsiderable, since the centre of gravity of the vessel lies in a vertical line to the left of B. But, since AB is greater than BK, $\tan \angle GBH$ must be greater than unity, and, θ being supposed to be greater than $\frac{\pi}{4}$, cot θ must be less than unity: thus the condition (1) is impossible. It is evident therefore that, whether the vessel be supposed to have weight or not, the plane may have any inclination up to 45°, but no greater.

2. The common intersection of two planes is a horizontal line, and between them rests an indefinitely thin hollow hemisphere, with its rim in contact with one of them, which is vertical, and the space between the internal superficies of the hemisphere and the vertical plane is filled with a homogeneous fluid: to find the pressure upon each of the planes and their greatest inclination for which the equilibrium is possible.

Let fig. (27) represent a section of the hemisphere and the two planes, made by a plane through the centre C of the sphere, perpendicular to the two planes; AE and AF denoting the sections of the two planes.

The plane AF will exert upon the hemisphere a force R, passing through C: also the fluid, which acts normally upon the hemisphere at every point, will produce upon it a resultant force P passing through C. The hemisphere is subject to the action of no other force beside these two, except the horizontal resultant S of the pressures of AE upon the rim of the hemisphere: the force S must therefore also pass through C.

Let r = the radius of the sphere, $\theta = \angle PCA$, $a = \angle EAF$. Then, for the equilibrium of the hemisphere, we have, resolving forces horizontally,

$$S + P \sin\theta = R \cos a \;\ldots\ldots\ldots\ldots (1),$$

and, resolving vertically,

$$P \cos\theta = R \sin a \ldots\ldots\ldots\ldots\ldots (2).$$

But $P \sin\theta$, $P \cos\theta$, will be equal respectively to the pressure of the fluid upon the vertical plane and the weight of the fluid, that is, to $\pi g\rho r^3$ and $\frac{2}{3}\pi g\rho r^3$. Hence the equations (1) and (2) become

$$S + \pi g\rho r^3 = R \cos a,$$

$$\tfrac{2}{3}\pi g\rho r^3 = R \sin a,$$

whence

$$R = \tfrac{2}{3}\pi g\rho r^3 \operatorname{cosec} a,$$

$$S = \tfrac{2}{3}\pi g\rho r^3 (\cot a - \tfrac{3}{2}).$$

Since S cannot have a negative value, the greatest value of a consistent with equilibrium is given by the equation

$$\cot a = \tfrac{3}{2}, \quad a = \tan^{-1}\tfrac{2}{3}.$$

3. A cylindrical vessel, the weight and thickness of which are inconsiderable, is placed with its base upon an inclined plane, and prevented from sliding by the roughness of the plane; to find the height to which it may be filled with fluid without oversetting.

If r be the radius of the cylinder, $\tan^{-1} a$ the inclination of the plane to the horizon, and c the length of the portion of the axis of the cylinder which is immersed in the fluid,

$$c = \frac{r}{a}\left\{1 + (1 - \tfrac{1}{2}a^2 - \tfrac{1}{4}a^4)^{\frac{1}{2}}\right\}.$$

Section II.

Stability of Vessels filled with Fluid.

1. A thin uniform hemispherical bowl of given weight, partly filled with fluid, is placed with its axis vertical upon

the highest point of a sphere: to ascertain the nature of the equilibrium in regard to stability, the bowl being supposed to be subject to a rolling but not to a sliding motion upon the supporting sphere.

Let A' (fig. 28) be the highest point of the supporting sphere, A the point of the bowl which, when it is in its position of equilibrium, is in contact with A'. Let O be the centre of the sphere of the bowl, O' that of the supporting sphere. Let Q be the point of contact when the bowl is displaced from its position of rest by a motion of rotation.

Let P denote the weight of the bowl and W of the fluid which it contains. Join OO' which will pass through Q, and produce OA to meet $O'A'$ produced in B. The centre of gravity G of the bowl, which is supposed to be of uniform thickness, will coincide with the middle point of OA, and the weight W of the fluid will act vertically downwards through O.

Let $OQ = r$, $O'Q = r'$, $\angle AOW = \theta$, $\angle O'OW = \psi$.

Then the moment of P and W about Q, tending to disturb the bowl still further from its position of equilibrium, is equal to

$$Wr \sin\psi + P\,(r \sin\psi - \tfrac{1}{2} r \sin\theta)$$
$$= (P + W)\, r \sin\psi - \tfrac{1}{2}\, Pr \sin\theta.$$

But, since the arc QA must be equal to the arc QA' by the nature of the displacement, which is due to rolling, we have

$$r'\psi = r\,(\theta - \psi),$$
$$\psi = \frac{r\theta}{r + r'}.$$

Hence the moment about Q is equal to

$$(P + W)\, r \sin\left(\frac{r\theta}{r + r'}\right) - \tfrac{1}{2} Pr \sin\theta.$$

In order that the bowl may tend to return to its position of rest, this expression must be negative: hence we must have

$$P > 2(P + W)\frac{\sin\left(\dfrac{r\theta}{r + r'}\right)}{\sin\theta}.$$

Suppose θ to be a very small angle: then approximately this inequality becomes

$$P > 2(P + W)\frac{r}{r + r'},$$

or

$$\frac{P}{W} > \frac{2r}{r' - r},$$

which is the condition for stability with regard to small displacements of rotation.

2. A cone of given density is scooped out so that the hollow part is the inscribed paraboloid, the vertex of which bisects the axis of the cone. To determine how much fluid of given density must be poured into it, that it may float in a state of neutral equilibrium when placed, with its vertex downwards, in another given fluid.

Let r denote the radius of the base of the cone, h its altitude; x the distance of the vertex A (fig. 29) of the paraboloid from the surface PQ of the fluid which it contains, x' the distance of the vertex O of the cone from the surface $P'Q'$ of the fluid in which the cone is immersed; V the volume of the fluid contained in the paraboloid and V' that of the fluid displaced by the immersion of the cone; y the radius of the circle PQ, y' that of the circle $P'Q'$; ρ the density of the solid substance of the cone, σ of the fluid poured into it, and σ' of the fluid in which it is immersed. Let H be the centre of gravity of the paraboloidal volume PAQ, H' of the conical volume $P'OQ'$; M the metacentre of the fluid PAQ and M' of the fluid displaced, $P'OQ'$. Then, in case of a slight angular displacement of the cone, the fluid PAQ will act on the solid portion of the cone vertically downwards through M, and the fluid $P'OQ'$ will act upon it vertically upwards through M'.

If the equilibrium be neutral, the moment of $V\sigma$ to twist the solid matter of the cone about its centre of gravity must be equal to that of $V'\sigma'$ to twist it in an opposite direction.

Let $\bar{x}$ denote the distance of the centre of gravity of the solid matter of the cone from its vertex: then, the volume of the whole cone KOL being $\frac{1}{3}\pi r^2 h$, the volume of the paraboloid KAL being $\frac{1}{4}\pi r^2 h$, the distance of the centre of gravity of the

whole cone *KOL* from *O* being $\frac{3}{4}h$ and that of the paraboloid *KAL* from *A* being $\frac{1}{3}h$, we have

$$\tfrac{1}{3}\pi r^2h\,.\,\tfrac{3}{4}h = \bar{x}\left(\tfrac{1}{3}\pi r^2h - \tfrac{1}{4}\pi r^2h\right) + \tfrac{1}{4}\pi r^2h\,.\left(\tfrac{1}{2}h + \tfrac{1}{3}h\right),$$

$$\tfrac{1}{4}h = \bar{x}\left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \tfrac{1}{4}\,.\tfrac{5}{6}h,$$

$$6h = 2\bar{x} + 5h, \quad \bar{x} = \tfrac{1}{2}h,$$

a result which shews that the centre of gravity of the solid portion of the cone coincides with the vertex *A* of the inscribed paraboloid.

Again, the equation to the generating parabola *KAL* being

$$y^2 = \frac{2r^2}{h}\,x, \qquad V.\,HM = \tfrac{1}{4}\pi y^4 = \frac{\pi r^4}{h^2}\,x^2,$$

or, since $\quad V = \int \pi y^2 dx = \dfrac{2\pi r^2}{h} \int x dx = \dfrac{\pi r^2 x^2}{h},$

$$HM = \frac{r^2}{h}.$$

But *AH* is equal to $\frac{2}{3}x$; hence

$$AM = \tfrac{2}{3}x + \frac{r^2}{h},$$

and therefore $\quad V.\,\sigma.\,AM = \sigma.\left(\dfrac{2}{3}\,x + \dfrac{r^2}{h}\right).\,\dfrac{\pi r^2 x^2}{h}.$

Again, $\quad V'.\,H'M' = \frac{1}{4}\pi y'^4,$

or, since V' is equal to $\frac{1}{3}\pi y'^2 x'$,

$$\tfrac{1}{3}x'.\,H'M' = \tfrac{1}{4}y'^2 = \frac{r^2x'^2}{4h^2},$$

$$H'M' = \frac{3}{4}\,.\,\frac{r^2x'}{h^2},$$

and therefore, *OA* being equal to $\frac{1}{2}h$ and *OH'* to $\frac{3}{4}x'$,

$$AM' = \frac{3}{4}\,\frac{h^2 + r^2}{h^2}\,x' - \tfrac{1}{2}h,$$

and $\quad \sigma'.\,V'.\,AM' = \frac{1}{3}\pi\sigma'\,\dfrac{r^2x'^3}{h^2}\,.\left(\dfrac{3}{4}\,\dfrac{h^2 + r^2}{h^2}\,x' - \tfrac{1}{2}h\right).$

Hence, the equilibrium being neutral,

$$V.\,\sigma.\,AM = V'.\,\sigma'.\,AM',$$

$$\sigma x^2.\,\left(\tfrac{2}{3}hx + r^2\right) = \sigma' x'^3\left(\frac{h^2 + r^2}{4h^2}\,x' - \tfrac{1}{6}h\right)\ldots\ldots(1).$$

Also, the weight of the fluid displaced being equal to the sum of the weights of the solid portion of the cone and the fluid poured into the paraboloid,

$$V.\,\sigma + (\text{vol. of cone} - \text{vol. of } KAL)\,\rho = V'.\,\sigma',$$

$$\frac{\pi\sigma r^2 x^2}{h} + \rho\left(\tfrac{1}{3}\pi r^2 h - \tfrac{1}{4}\pi r^2 h\right) = \tfrac{1}{3}\pi\sigma'\,\frac{r^2 x'^3}{h^2},$$

$$\sigma x^2 + \tfrac{1}{12}\rho h^2 = \frac{\sigma' x'^3}{3h} \,\ldots\ldots\ldots\ldots\ldots\,(2).$$

The value of x is determined by the two equations (1) and (2).

CHAPTER IX.

TENSION OF VESSELS CONTAINING FLUID.

LET p be the pressure exerted at any point of a vessel of any form by the fluid which it contains; r, s, its principal radii of curvature at the point; and u, v, the respective tensions of the material of the vessel in the corresponding principal sections. Then

$$p = \frac{u}{r} + \frac{v}{s}.$$

If the vessel is immersed in fluid, p must be here taken to represent the difference of the pressures of the exterior and interior fluids.

If the tensions be the same in both principal sections, u, v, being each denoted by the same letter t, we have

$$p = t\left(\frac{1}{r} + \frac{1}{s}\right),$$

and, if $s = r$, as in a spherical envelop,

$$p = \frac{2t}{r}.$$

Supposing the vessel to be cylindrical, one of the principal radii of curvature will be infinite: putting $s = \infty$, we have

$$p = \frac{u}{r},$$

a relation between the pressure, the tension, and the radius of curvature, the discovery of which is due to James Bernoulli.

See Jacobi Bernoulli Opera, tom. II. p. 1043, *de Curvatura Fili ab innumeris potentiis extensi*.

1. A tube, every horizontal section of which is circular, is filled with fluid which is acted on by gravity, the axis of

the tube being vertical: to find the law of the thickness of the tube that it may be, in regard to the stress of the fluid, of the same strength throughout.

Let t denote the tension of any circular section of the bore of which r is the radius, and p the pressure of the fluid at each point of this section. Then

$$t = pr \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let h represent the depth of this section below the surface of the fluid, g the force of gravity, ρ the density of the fluid: then

$$p = g\rho h,$$

and therefore, from (1), $\quad t = g\rho hr \ldots\ldots\ldots\ldots\ldots\ldots (2).$

Suppose τ to denote the thickness of the tube at the section, and f the force exerted by a unit of thickness, which, by the condition of the problem, will be the same throughout the tube. Then

$$t = f\tau,$$

and therefore, from (2), $\quad f\tau = g\rho hr,$

$$\tau \propto hr.$$

If the tube be of uniform bore, its thickness must vary from section to section as the depth below the surface of the fluid.

Encycl. Metrop. Mixed Sc. vol. I. p. 177.

2. A given mass of known elastic fluid is confined in a slightly elastic spherical envelop of given material, the natural radius of which is assigned. Supposing the tension of the envelop to be as the linear extension, to determine approximately the increment of its radius.

Let r be the radius of the expanded envelop, R its natural radius, p the pressure of the fluid at each of its points. Then, t being the tension,

$$pr = 2t \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let μ denote the modulus of elasticity of the material of the envelop; then $\quad r - R = \mu Rt \ldots\ldots\ldots\ldots\ldots (2).$

From (1) and (2), since μ is, by the condition of problem, supposed to be small, we have

$$r - R = \tfrac{1}{2}\mu R^2 p \ldots\ldots\ldots\ldots\ldots (3).$$

Let k represent the elasticity of the fluid, supposing a unit of its mass to be compressed into a unit of volume; then, $\frac{4}{3}\pi r^3$ being the volume of the envelop, and M being taken to represent the mass of the fluid in the envelop,

$$p = k \cdot \frac{M}{\frac{4}{3}\pi r^3} = \frac{3kM}{4\pi R^3}, \text{ nearly}:$$

and therefore, from (3),

$$r - R = \frac{3\mu k}{8\pi} \cdot \frac{M}{R}.$$

3. A paraboloid is filled with fluid, the pressure at any point of which is equal to p; to determine the tension at this point, its distance from the focus being r, the tension in both principal sections being supposed to be the same.

The radius of curvature of the generating curve is equal to

$$\frac{2r^{\frac{3}{2}}}{m^{\frac{1}{2}}},$$

and the radius of curvature of the other principal section, $y^2 = 4mx$ being the equation to the generating parabola, will be, if θ be the angle between the ordinate and the normal,

$$y \sec \theta = y\left(1 + \frac{4m^2}{y^2}\right)^{\frac{1}{2}} = (y^2 + 4m^2)^{\frac{1}{2}} = 2\,(mx + m^2)^{\frac{1}{2}} = 2\,(mr)^{\frac{1}{2}}.$$

Hence r', s', denoting the two radii of curvature,

$$p = t\left(\frac{1}{r'} + \frac{1}{s'}\right) = t\left\{\frac{m^{\frac{1}{2}}}{2r^{\frac{3}{2}}} + \frac{1}{2\,(mr)^{\frac{1}{2}}}\right\}$$

$$= t \cdot \frac{m + r}{2m^{\frac{1}{2}} r^{\frac{3}{2}}},$$

$$t = p \cdot \frac{2m^{\frac{1}{2}} r^{\frac{3}{2}}}{m + r}.$$

4. To find the form of a rectangle of cloth which, having two opposite sides supported parallel to each other in a horizontal plane, is pressed by the weight of a fluid contained in it, the fluid being supposed to be prevented running out from the

cloth by two vertical boards in contact with its two other sides.

Let x, y, be horizontal and vertical coordinates of any point in a vertical section of the cloth at right angles to the two fixed ends, the axis of y extending vertically downwards.

Then, t denoting the tension at this point, and r the radius of curvature,

$$t = pr:$$

but

$$p = g\rho y:$$

hence

$$g\rho y = \frac{t}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But since the fluid acts normally on the cloth at every point, it follows that

$$t = c,$$

c being a constant quantity: hence, from (1), we see that

$$g\rho y = \frac{c}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now supposing s, the arc, to be the independent variable, we know that

$$r = \frac{dy\,ds}{d^2x};$$

we have therefore, from (2),

$$g\rho y = c\,\frac{d^2x}{dy\,ds},$$

or, putting $c = \frac{1}{2}g\rho a^2$,

$$y = \tfrac{1}{2}a^2\,\frac{d^2x}{dy\,ds}.$$

Integrating, ds being considered constant, we get

$$y^2 = a^2\,\frac{dx}{ds},$$

no constant being added, if the origin be so chosen that y and $\frac{dx}{ds}$ are zero simultaneously.

Since ds^2 is equal to $dx^2 + dy^2$, we have

$$y^4\,(dx^2 + dy^2) = a^4\,dx^2,$$

$$dx = \frac{y^2\,dy}{(a^4 - y^4)^{\frac{1}{2}}},$$

the differential equation to the curve.

This curve is called the Lintearia in consequence of the mode of its formation. Its equation shews that it coincides with the Elastica.

The history of the discovery of the Lintearia is given in the chapter on Resistances, after the solution of the problem of the Velaria.

5. A homogeneous incompressible fluid completely enclosed in a thin and perfectly extensible envelop, which exercises a uniform tension in every direction in its surface, at any point, is placed upon a horizontal plane: to find the differential equation to a meridian of the surface (of revolution) assumed by the fluid.

Let EAE (fig. 30) be the intersection of the plane of a meridian $AEBEA$ with the horizontal plane, AB being the axis of the envelop. Draw PM at right angles to AB from any point P of the meridian. Let $AB = a$, $AM = y$, $PM = x$, $T =$ the tension at each point of the surface, which will be the same throughout, the action of the fluid on the surface being normal to it at each point. Let r, s, be the principal radii of curvature of the envelop at P, r being the radius of curvature in the meridional plane: let R be the radius of curvature at B. Let p represent the pressure of the fluid at P.

Then, for the equilibrium, there is

$$p = T\left(\frac{1}{r} + \frac{1}{s}\right),$$

or, the pressure at B being $\frac{2T}{R}$,

$$\frac{2}{R} + \frac{g\rho}{T}(a - y) = \frac{1}{r} + \frac{1}{s}.$$

But
$$\frac{1}{r} = -\frac{\frac{d^2x}{dy^2}}{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}},$$

and, θ denoting the inclination of PM to the normal at P,

$$\frac{1}{s} = \frac{\cos\theta}{x} = \frac{1}{x(1 + \tan^2\theta)^{\frac{1}{2}}} = \frac{1}{x\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}}.$$

$$\text{Hence } \frac{2}{R} + \frac{g\rho}{T}(a - y) = -\frac{\dfrac{d^2x}{dy^2}}{\left(1 + \dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}} + \frac{1}{x\left(1 + \dfrac{dx^2}{dy^2}\right)^{\frac{1}{2}}},$$

which is the differential equation to the meridian.

COR. When the quantity of fluid is very great, so that it covers a very large portion of the horizontal plane, the differential equation may be once integrated, the result being

$$T\left\{1 - \frac{1}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}}\right\} = \tfrac{1}{2}g\rho(2ay - y^2),$$

where a, the value of y for points at a considerable distance from the edge of the fluid, where the curve is sensibly a horizontal straight line, will be equal to $\left(\dfrac{4T}{g\rho}\right)^{\frac{1}{2}}$.

6. A cylindrical pipe of given radius is formed with metal of a given thickness: given the greatest weight which a cylindrical wire of the same material and thickness can support: to find the greatest height of fluid which the pipe can sustain without bursting.

Let w denote the greatest weight which can be sustained by the wire, of which the breadth is supposed to be unity: let h denote the required height of the fluid, r the radius of the pipe, and ρ the density of the fluid. Then

$$h = \frac{w}{g\rho r}.$$

7. To determine the ratio of the thicknesses of two vertical cylindrical vessels containing fluids which are just on the point of bursting them.

If t, t', denote the tenacities of the materials of the two vessels, ρ, ρ', the densities of the fluids which they contain, h, h', the depths of the fluids in the two vessels, d, d', the diameters of their bases, and E, E', the thicknesses of the vessels, we shall have

$$E : E' :: \frac{\rho h d}{t} : \frac{\rho' h' d'}{t'}.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 44.

CHAPTER X.

GENERAL EQUATIONS OF THE EQUILIBRIUM OF FLUIDS.

In the preceding chapters we have been concerned only with that class of hydrostatical problems in which gravity is supposed to be the sole acting force. We shall devote this chapter to the discussion of various problems in which a fluid is considered to be subject to the action of any system of forces whatever.

Let X, Y, Z, represent the sum of the components, resolved parallel to any system of rectangular axes, of all the accelerating forces acting at any point of a fluid: let p represent the unit of pressure and ρ the density at this point. Then the functionality subsisting between the pressure, density, and accelerating forces, is expressed by the equation

$$dp = \rho\,(Xdx + Ydy + Zdz),$$

or, supposing R to be the resultant of X, Y, Z, and dr to be the diagonal of the parallelopiped, of which the three contiguous edges are dx, dy, dz,

$$dp = \rho Rdr.$$

At a free surface or one of equal pressure, dp must evidently be zero: hence $$Xdx + Ydy + Zdz = 0,$$

or $$Rdr = 0.$$

The general problem of the form and circumstances of the equilibrium of a fluid acted on by any system of forces, began about the time of Newton to excite great interest in the minds of mathematicians, owing to its connection with the particular problem of the forms of the earth and the planets, on the hypothesis of their original fluidity. Huyghens, in treating on this matter, has assumed, as his principle of equilibrium, the

perpendicularity of gravity to the surface. Newton* has availed himself of the principle of the equilibrium of the central columns of the fluid. Bouguer† and Maupertuis have shewn that it was necessary to equilibrium that the principles of Huyghens and Newton should coexist. Maclaurin has added a new condition, viz. that any canal formed of two rectilinear branches, commencing at the surfaces and terminating at any point whatever in the fluid, be in equilibrium. Clairaut, generalizing the conceptions of Newton and Maclaurin, has adopted, as the basis of his reasonings on the equilibrium of fluids, these two principles.

(1). *Une masse de fluide ne saurait être en équilibre, que les efforts de toutes les parties qui sont comprises dans un canal de figure quelcônque qu' on imagine traverser la masse entière, ne se détruisent mutuellement.*

(2). *Afin qu' une masse de fluide puisse être en équilibre, il faut que les efforts de toutes les parties de fluide renfermées dans un canal quelconque rentrant en lui-même, se détruisent mutuellement.*

The discovery of the general equation

$$dp = \rho\,(Xdx + Ydy + Zdz)$$

for the pressure at any point of a fluid in equilibrium, under the action of any system of forces whatever, is due to Clairaut, whose researches on this subject were published in his *Théorie de la Figure de la Terre*, in the year 1743.

The fundamental principles of Clairaut were shewn by D'Alembert, in his *Essai sur la Résistance des Fluides*, Paris 1752, art. 18, to result as a necessary consequence from the principle of Maclaurin. D'Alembert‡ observes also that Daniel Bernoulli had already laid down substantially the same principle in his *Hydrodynamique, section seconde*, § 3 p. 18, in the following words. "*In aquâ stagnante tubus utcumque formatus fingi potest, in quo utique aqua situm servabit quem*

* Principia, Lib. III. prop. 19.

† *Mémoires de l' Académie de Paris*, 1733.

‡ *Traité des Fluides*, p. 49.

anteà habuit, cum perinde sit, sive aqua tubo inclusa coerceatur lateribus tubi, sive circumstagnante aquâ." The formula of Clairaut received an elegant demonstration by Euler, who conceived the fluid as consisting of an indefinite number of indefinitely small parallelopipeds, and expressed the conditions for the equilibrium of any one of these elements of the volume: this method of demonstration has been adopted in almost all the modern treatises on hydrostatics. Euler's investigations on this subject were published in the *Mémoires de l'Académie de Berlin* for the year 1755. The demonstration of the equation of fluid equilibrium has been effected by Lagrange in his *Traité de Mécanique Analytique* without assuming any property of matter peculiar to fluids, which he treated as consisting of an indefinite number of small molecules freely moveable among each other. For additional information on the history of the development of the science of the equilibrium of fluids, the reader is referred to Lagrange's *Traité de Mécanique Analytique* and to Montucla's *Histoire des Mathématiques*, tom. IV. p. 178, &c.

SECTION I.

Incompressible Fluids.

1. A mass of incompressible fluid is acted on by an attractive force varying directly as the distance from a fixed centre: to find the form of the surface of the fluid in equilibrium and the pressure at any point within it, the surface of the fluid being free from pressure.

Let ρ be the density of the fluid, p the pressure at any point at a distance r from the centre of force: then, taking μ to denote the absolute force of attraction,

$$dp = -\mu\rho r\,dr.$$

Integrating, we have

$$p = C - \tfrac{1}{2}\mu\rho r^2:$$

but, at the surface of the fluid, $p = 0$; hence the equation to the surface is

$$C = \tfrac{1}{2}\mu\rho r^2.$$

The form of the surface is therefore spherical, the centre of force being the centre of the sphere. Let a denote its radius; then

$$C = \tfrac{1}{2}\mu\rho a^2,$$

and therefore

$$p = \tfrac{1}{2}\mu\rho\,(a^2 - r^2).$$

At the centre of the sphere of fluid,

$$p = \tfrac{1}{2}\mu\rho a^2.$$

This problem would correspond to the investigation of the form of the earth supposed to consist entirely of fluid and to be devoid of its motion of rotation.

Euler: *Mémoires de l'Acad. des Sciences de Berlin,* 1755, p. 258.

2. A fluid mass, the particles of which attract with a force varying inversely as the square of the distance, arranges itself in concentric layers about a homogeneous solid sphere, of which the molecules attract according to the same law: supposing the relation between the pressure and the density to be expressed by the equation

$$p = n\rho^2 + a,$$

n and a being constants, to determine the law of the density.

The concentric fluid shells, which are external to any particle, will exercise no attraction upon it, and those shells which are between the particle and the solid sphere will attract it just as if they were condensed into an intense particle at the centre of the sphere.

Now, the radius of the solid sphere being denoted by a, the mass of the fluid between the solid sphere and a sphere of which the radius is r, will be equal to

$$\iint r\,d\theta\,dr\,.\,2\pi r \sin\theta\,.\,\rho$$

$$= 2\pi \int_0^\pi\int_a^r \rho r^2 \sin\theta\,dr\,d\theta$$

$$= 4\pi \int_a^r \rho r^2\,dr.$$

Hence, the mass of the solid sphere being M, we shall have for the united attraction of the solid sphere, and of a shell

of fluid, of which the radii are a, r, upon a particle of fluid at a distance r from the centre,

$$\frac{1}{r^2}\left\{4\pi c\int_a^r \rho r^2 dr + c'M\right\}\ldots\ldots\ldots\ldots(1),$$

c, c', being the absolute forces of attraction of units of mass of the fluid and solid respectively, condensed into points.

But, by the general equation of fluid equilibrium,

$$dp = -\rho R dr,$$

and therefore, by the condition of the problem,

$$2n\rho\, d\rho = -\rho R dr,$$

or $$2n\, d\rho = -R dr:$$

hence, by (1), $$2n d\rho = -\frac{dr}{r^2}\left\{4\pi c\int_a^r \rho r^2 dr + c'M\right\},$$

or, multiplying both sides of the equation by r^2, and then differentiating,

$$2nr^2\frac{d^2\rho}{dr^2} + 4nr\frac{d\rho}{dr} + 4\pi c\rho r^2 = 0,$$

$$r\frac{d^2\rho}{dr^2} + 2\frac{d\rho}{dr} + h^2 r\rho = 0,$$

h being a constant.

The integral of this equation is

$$\rho = \frac{A\sin(hr + c)}{r},$$

A, h, c, being constants; the law of the density is therefore determined.

3. A hollow sphere is filled with a homogeneous gravitating fluid, and a centre of force, attracting directly as the distance, resides in the upper extremity of its vertical diameter: to compare the whole normal pressure on the upper and lower hemisphere, the pressure at the highest point of the sphere being supposed to be zero.

Let OO' (fig. 31) be the vertical diameter, and C the centre of the sphere. Take P any point in the surface of the sphere; join OP, CP: let $\angle PCO' = \theta$, $OP = r$, $OC = a$. Then, μ being the

absolute force, x the vertical distance of P below O, and p the pressure at P,

$$dp = \rho (g\,dx - \mu r\,dr),$$

$$p = \rho (gx - \tfrac{1}{2}\mu r^2) = \rho (gx - \mu a x) = \rho (g - \mu a)(a + a \cos \theta),$$

no constant being added because $p = 0$ when $x = 0$ and $r = 0$.

The pressure on the lower hemisphere is equal to

$$\int_0^{\frac{1}{2}\pi} a\,d\theta \,.\, 2\pi a \sin \theta \,.\, p = 2\pi a^2 \int_0^{\frac{1}{2}\pi} p \sin \theta\, d\theta$$

$$= 2\pi a^3 \rho (g - \mu a) \int_0^{\frac{1}{2}\pi} (1 + \cos \theta) \sin \theta\, d\theta$$

$$= 3\pi a^3 \rho (g - \mu a).$$

The pressure on the upper hemisphere is equal to

$$2\pi a^3 \rho (g - \mu a) \int_{\frac{1}{2}\pi}^{\pi} (1 + \cos \theta) \sin \theta\, d\theta$$

$$= \pi a^3 \rho (g - \mu a).$$

Hence the former pressure is three times as great as the latter.

COR. 1. If $\mu = \dfrac{g}{a}$, we see that $p = 0$: the fluid would therefore remain in equilibrium in its spherical form if the spherical envelop were removed.

COR. 2. If the force were repulsive instead of attractive, μ should be replaced by $-\mu$: thus

$$p = \rho (g + \mu a),$$

and, if $\mu = \dfrac{g}{a}$, the whole normal pressure on the spherical envelop would be the same as though the force did not exist and the intensity of gravity were doubled.

4. A mass of fluid is acted on by forces parallel to three rectangular axes, which vary respectively, for each particle, directly as its perpendicular distance from the planes

$$lx + my + nz = 0, \quad mx + m'y + n'z = 0, \quad nx + n'y + n''z = 0,$$

where (l, m, n), (m, m', n'), (n, n', n''), are the direction-cosines of the three planes respectively: to determine the form of the free surface.

Since $p = 0$ at every point of a free surface, we have

$$X dx + Y dy + Z dz = 0 \; \ldots\ldots\ldots\ldots (1).$$

Now, x, y, z, being the coordinates of any particle of fluid, we have, by the conditions of the problem,

$$X=\mu(lx+my+nz),\quad Y=\mu(mx+m'y+n'z),\quad Z=\mu(nx+n'y+n''z),$$

μ being some constant quantity.

Hence the differential equation to the free surface will be

$$(lx+my+nz)\,dx+(mx+m'y+n'z)\,dy+(nx+n'y+n''z)\,dz=0,$$

the integral of which is evidently

$$lx^2+m'y^2+n''z^2+2n'yz+2nzx+2mxy=C,$$

C being a constant quantity. The free surface is therefore a surface of the second order.

5. A given quantity of an incompressible fluid, each particle of which is attracted towards a fixed centre by a force which varies as the distance, is separated from it by a given fixed plane: to determine the pressure which the fluid exerts upon the plane.

Let r denote the distance of any particle of the fluid from the fixed centre of force, and μ the absolute force of attraction. Then, p denoting the pressure at this particle,

$$dp=-\mu\rho\,.\,r\,dr,$$

whence

$$p=C-\tfrac{1}{2}\mu\rho r^2,$$

or, putting

$$C=\tfrac{1}{2}\mu\rho a^2,$$

$$p=\tfrac{1}{2}\mu\rho(a^2-r^2).$$

At the free surface of the fluid, $p=0$, and therefore the free surface is a sphere, the radius of which is a, its centre being at the centre of force.

Let s denote the distance of any point in the fixed plane from the centre of the circular section of the sphere made by the plane. Then, c denoting the distance of the centre of attraction from the fixed plane, the whole pressure of the fluid on the plane will be equal to

$$\begin{aligned}&\int p\,.\,2\pi s\,ds\\ &=\mu\pi\rho\int(a^2-r^2)\,s\,ds\\ &=\mu\pi\rho\int(a^2-c^2-s^2)\,s\,ds,\end{aligned}$$

the integration being effected from $s=0$ to $s=(a^2-c^2)^{\frac{1}{2}}$.

Hence the pressure will be equal to

$$\pi\mu\rho\,\{\tfrac{1}{2}(a^2-c^2)^2-\tfrac{1}{4}(a^2-c^2)^2\}$$
$$=\tfrac{1}{4}\pi\mu\rho\,(a^2-c^2)^2.$$

Let C represent the given volume of the fluid: then, θ denoting the inclination of any radius of the spherical surface to the distance c,

$$C=\int\pi a^2\sin^2\theta\,.\,d\,(a\cos\theta),\quad\text{from }\cos\theta=\frac{c}{a},\text{ to }\cos\theta=1,$$
$$=\pi a^3\int(1-\cos^2\theta)\,d\cos\theta$$
$$=\pi a^3\,(\cos\theta-\tfrac{1}{3}\cos^3\theta)$$
$$=\pi a^3\left(\frac{2}{3}-\frac{c}{a}+\frac{1}{3}\frac{c^3}{a^3}\right)$$
$$=\pi\,(\tfrac{2}{3}a^3-a^2c+\tfrac{1}{3}c^3):$$

the value of a will therefore depend upon the solution of a cubic equation.

6. The annular space contained between two cylinders having the same axis, and two planes perpendicular to this axis, is filled with a homogeneous fluid, and is acted on by a system of forces such that

$$X=-\frac{y}{x^2+y^2},\quad Y=\frac{x}{x^2+y^2},\quad Z=0,$$

the axis of z being that of the cylinders. If a rigid plane or surface of any form be placed in the annular space, so as to stop the passage round, to determine the difference of the pressures on the two sides of this surface at any point.

For equilibrium we have the equation

$$dp=\rho\,(Xdx+Ydy+Zdz)$$
$$=\rho\left(\frac{xdy-ydx}{x^2+y^2}\right):$$

putting $x=r\cos\theta$, $y=r\sin\theta$, this equation becomes

$$dp=\rho d\theta;$$

which shews that dp is equal to a perfect differential, a con-

dition sufficient for the possibility of the equilibrium of the fluid. Integrating from $\theta = \theta$ to $\theta = 2\pi + \theta$, and representing by p', p'', the pressures on the two sides of the interposed surface at any point, we have

$$p'' - p' = 2\pi\rho.$$

7. A mass of incompressible fluid is acted on by a central force attracting according to the inverse square of the distance: to find the form of the surface, when the fluid is at rest, and the pressure at any point within the fluid; the surface of the fluid being free from pressure.

The surface will be a sphere concentric with the centre of force, and, if the volume of the fluid be such that the radius of the sphere is equal to a, we shall have, p denoting the pressure at a distance r from the centre, μ the absolute force of attraction and ρ the density of the fluid,

$$p = \mu\rho\left(\frac{1}{r} - \frac{1}{a}\right).$$

Euler: *Mémoires d l'Acad. des Sciences de Berlin,* 1755, p. 258.

8. To determine the form of the free surface of a given volume of incompressible fluid, each of its particles being acted on by a force the components of which, parallel to the axes of a system of rectangular coordinates, are

$$-\frac{\mu x}{a^2}, \quad -\frac{\mu y}{b^2}, \quad -\frac{\mu z}{c^2}.$$

If V represent the given volume, the required equation will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \lambda^2,$$

λ being known from the equation

$$\lambda^3 = \frac{3V}{4\pi abc}.$$

9. A mass of incompressible fluid is acted upon by forces parallel to three rectangular axes: the force at the point x, y, z, parallel to the axis of x, is equal to $k(y + z)$, those parallel to

the axes of y, z, respectively, being $k(z+x)$, $k(x+y)$: to determine the equation to the free surface of the fluid.

The equation to the surface will be

$$yz + zx + xy = c^2,$$

c being a constant quantity.

Section II.

Elastic Fluids.

1. A solid globe is placed centrally in a globular cavity, and the space between the surfaces is filled with a mass of a known elastic fluid, every particle of which is attracted towards the centre of the globe by a force proportional to the inverse square of the distance: supposing the fluid to be in equilibrium, and the absolute force of attraction to be given, to compare its densities at the surface of the globe and the boundary of the cavity.

Let μ denote the absolute force of attraction, and k the ratio of the density to the pressure at each point of the fluid: then

$$dp = -\mu\rho\frac{dr}{r^2},$$

$$d\rho = -k\mu\rho\frac{dr}{r^2},$$

$$\log\rho = C + \frac{k\mu}{r}.$$

Let δ represent the density at the surface of the globe, and δ' that at the boundary of the cavity. Then, a, a', denoting the radii of the globe and cavity respectively,

$$\log\delta = C + \frac{k\mu}{a},$$

$$\log\delta' = C + \frac{k\mu}{a'},$$

and therefore $$\frac{\delta}{\delta'} = \varepsilon^{k\mu\left(\frac{1}{a} - \frac{1}{a'}\right)}.$$

COR. Let M represent the mass of the fluid: then

$$M = \int_a^{a'} \rho \,.\, 4\pi r^2 dr = 4\pi \int_a^{a'} \delta \varepsilon^{k\mu\left(\frac{1}{r} - \frac{1}{a}\right)} . r^2 dr,$$

$$\delta = \frac{M\varepsilon^{\frac{k\mu}{a}}}{4\pi \int_a^{a'} \varepsilon^{\frac{k\mu}{r}} r^2 dr};$$

and similarly $$\delta' = \frac{M\varepsilon^{\frac{k\mu}{a'}}}{4\pi \int_a^{a'} \varepsilon^{\frac{k\mu}{r}} r^2 dr};$$

which two formulæ, supposing the mass of the fluid to be given, determine the absolute values of the densities δ and δ'.

2. A fluid is in equilibrium under the action of a force attracting towards a fixed centre with an intensity varying inversely as the distance: the density being supposed to vary as the pressure, to find the density at any point within the fluid.

Let $p = \mu k \rho$, and $\frac{\mu}{r}$ = the central force: then

$$dp = -\rho \frac{\mu}{r} dr,$$

and therefore $$k \frac{d\rho}{\rho} = -\frac{dr}{r}:$$

hence, a denoting a constant quantity,

$$k \log \rho = \log \frac{a}{r},$$

$$\rho^k = \frac{a}{r},$$

$$\rho = \left(\frac{a}{r}\right)^{\frac{1}{k}}.$$

Let ρ' and r' denote any other corresponding values of ρ and r:

then $$\rho' = \left(\frac{a}{r'}\right)^{\frac{1}{k}},$$

$$\frac{\rho}{\rho'} = \left(\frac{r'}{r}\right)^{\frac{1}{k}}.$$

Suppose that $r' = cr$, c being a constant quantity: then

$$\frac{\rho}{\rho'} = c^{\frac{1}{k}}, \quad \rho' = \rho c^{-\frac{1}{k}},$$

a result which shews that, for a series of values of r in geometrical progression, the corresponding series of values of ρ will also be in geometrical progression.

Newton: *Principia*, lib. II. sect. 5, Prop. 21.
Herman: *Phoronomia*, p. 202.

3. A fluid, the density of which is always proportional to the pressure, and of which the particles are attracted towards a fixed centre with forces varying inversely as the square of the distance, is in equilibrium: to determine the density at any point of the fluid.

Let $p = k\rho$, and $\frac{\mu}{r^2}$ = the central force: then

$$dp = -\rho \frac{\mu}{r^2} dr,$$

$$k \frac{d\rho}{\rho} = -\frac{\mu}{r^2} dr,$$

whence $$\log \frac{\rho}{c} = \frac{\mu}{kr},$$

c being a constant quantity; or

$$\rho = c\, e^{\frac{\mu}{kr}}.$$

Cor. Let r' be any other distance corresponding to a density ρ'; and let $$\frac{1}{r'} - \frac{1}{r} = \frac{1}{\beta},$$

β being a constant: then

$$\rho' = c\, e^{\frac{\mu}{kr'}},$$

$$\frac{\rho'}{\rho} = e^{\frac{\mu}{k\beta}},$$

or, if the distances be taken in musical progression, the corresponding densities of the fluid will be in geometrical progression.

Newton: *Principia*, lib. II. sect. 5, Prop. 22.
Brook Taylor: *Methodus Incrementorum*, Prop. 26, p. 103.
Herman: *Phoronomia*, p. 202.

4. A fluid, the density of which at any point varies as the n^{th} power of the pressure, is attracted by a force tending towards a fixed centre and varying according to any law of the distance: to determine the density at any point of the fluid.

Let ρ be the density and F the force of attraction at any point at a distance x from the centre of force. Then

$$dp = -\rho F dx,$$

but, c being some constant quantity,

$$\rho = c^n p^n, \quad \rho^{\frac{1}{n}} = cp;$$

hence $$d\,(\rho^{\frac{1}{n}}) = -c\rho F dx,$$

$$\frac{1}{n}\rho^{\frac{1}{n}-2}\, d\rho = -cF dx,$$

or, integrating, $$\frac{1}{1-n}\rho^{\frac{1-n}{n}} = q - c\int F dx,$$

q being an arbitrary constant, the value of which is dependent upon the limits of the fluid.

Varignon has solved this problem in a slightly different manner. In fact the pressure of the superincumbent column of fluid at any point is equal to

$$-\int \rho F dx,$$

the limit of the integration being defined by the boundary of the fluid.

Hence, on the hypothesis that the density varies as the n^{th} power of the weight of the superincumbent column at any point,

$$\rho = c^n \{-\int \rho F dx\}^n,$$

$$\rho^{\frac{1}{n}} = -c\int \rho F dx;$$

whence, differentiating,

$$\frac{1}{n}\rho^{\frac{1}{n}-1}\, d\rho = -c\rho F dx,$$

$$\frac{1}{n}\rho^{\frac{1}{n}-2}\, d\rho = -cF dx,$$

or, integrating,
$$\frac{1}{1-n}\rho^{\frac{1-n}{n}} = q - c\int F dx,$$

the same equation as we obtained above.

Cor. Let $n = 1$, and $F = \mu x$; then, putting

$$q = q' + \frac{1}{1-n},$$

q' being any constant, we have

$$\frac{1}{1-n}\left(\rho^{\frac{1-n}{n}} - 1\right) = q' - \mu c\int x\, dx$$

$$= q' - \tfrac{1}{2}\mu c x^2.$$

But, since $n = 1$,

$$\frac{0}{0} = \frac{\rho^{\frac{1-n}{n}} - 1}{1-n} = \frac{-\frac{1}{n^2}\log \rho \,.\, \rho^{\frac{1-n}{n}}}{-1} = \log \rho:$$

hence
$$\log \rho = q' - \tfrac{1}{2}\mu c x^2.$$

Let ρ' be the density at any distance x'; then

$$\log \rho' = q' - \tfrac{1}{2}\mu c x'^2:$$

or
$$\log \frac{\rho}{\rho'} = \tfrac{1}{2}\mu c\,(x'^2 - x^2).$$

Varignon: *Mémoires de l'Académie des Sciences de Paris*, 1716, p. 108.

5. A fluid, the density of which is always proportional to the pressure, is acted on by forces tending to a centre and varying directly as the distance: to determine the law of the density.

Let ρ, ρ', be the densities at distances r, r', from the centre: then

$$2k \log \frac{\rho}{\rho'} = r'^2 - r^2,$$

which shews that, if the squares of the distances be taken in arithmetical progression, the densities will be in geometrical progression.

Newton: *Principia*, lib. II. sect. 5, Prop. 22. *Scholium.*
Herman: *Phoronomia*, p. 202.

6. A fluid, the density of which is always proportional to the pressure, is acted on by gravity considered as invariable at all altitudes: to determine the law of the density.

If g denote the force of gravity, and ρ, ρ', the densities at altitudes r, r', respectively,

$$k \log \frac{\rho}{\rho'} = g\,(r' - r),$$

which shews that, as the altitudes increase in arithmetical, the densities will decrease in geometrical progression.

Halley: *Philosophical Transactions.*
Newton: *Principia*, lib. II. sect. 5, Prop. 22. *Scholium.*
Herman: *Phoronomia*, p. 202.
Varignon: *Mémoires de l'Académie des Sciences de Paris*, 1716, p. 118.

7. The elasticity of the air being supposed to vary as the $\frac{m+1}{m}$th power of the density, to find the ratio of the altitude of the atmosphere above the surface of the earth to the altitude of the atmosphere supposed homogeneous, the pressure of the atmosphere at the surface of the earth being supposed the same in both hypotheses.

If h represent the altitude of the homogeneous and h' of the heterogeneous atmosphere, then, approximately,

$$h' = (m + 1)\,h,$$

h being supposed to be small in comparison with the earth's radius.

SECTION III.

Equilibrium of Revolving Fluids.

If a volume of fluid, invariable in form, and of which the molecules remain at rest in relation to its superficies, revolve with a uniform angular velocity ω about a fixed axis under the action of any system of forces, each of which is either parallel to this axis or tends towards a fixed point within it, the relation between the pressure, density, and impressed forces will be expressed in rectangular coordinates by the equation

$$dp = \rho\,(Xdx + Ydy + Zdz + \omega^2 rdr),$$

r representing the distance of any molecule from the axis of revolution.

This formula is given by Clairaut in his *Théorie de la Figure de la Terre*, p. 101, *seconde edition.*

1. An indefinitely thin tube, of which one branch AB (fig. 32) is horizontal and the other BC vertical, turns with a given uniform velocity about a vertical axis AD; the portion PBQ of the tube is occupied by incompressible fluid, which remains in equilibrium during the motion of the tube: to find the relation between the lengths BP and AQ.

Let p denote the pressure at any point within the fluid PBQ; ρ the density of the fluid, ω the angular velocity about AD, x the distance of any point of the fluid PBQ from AD, y the perpendicular distance of any point in the fluid from AB. Then for equilibrium we have

$$dp = \rho\,(\omega^2 xdx - gdy);$$

integrating we obtain

$$p = \rho\,(\tfrac{1}{2}\omega^2 x^2 - gy) + C;$$

but, at Q, $p = 0$, $y = 0$; hence, putting $AQ = b$,

$$0 = \tfrac{1}{2}\rho\omega^2 b^2 + C;$$

also, at P, $p = 0$; hence, denoting AB by a, and putting $BP = h$, we get

$$0 = \rho\,(\tfrac{1}{2}\omega^2 a^2 - gh) + C.$$

We have therefore

$$0 = \tfrac{1}{2}\omega^2 (a^2 - b^2) - gh,$$

$$h = \frac{a^2 - b^2}{2g}\, \omega^2.$$

Ducret: *Essais sur les Machines Hydrauliques*, p. 229. Paris, 1777.

2. A solid cylinder floats in a fluid in a cylindrical vessel, with its axis vertical and coincident with that of the vessel, and the system revolves about the common axis with a given angular velocity ω: to determine the difference between the altitude of the cylinder under such circumstances and the altitude which it would have were the fluid at rest.

Let $ABCD$ (fig. 33) be the floating cylinder, $A'\alpha\beta B'$ a section of the surface of the fluid made by a plane through the axis KL of the hollow cylinder $A'D'C'B'$. Let O be the point in which the curve, of which $A'\alpha$, $B'\beta$, are portions, meets the axis KL. Let ω represent the angular velocity, x the altitude above O of any point in the free surface of the fluid, and y the distance of this point from the axis OL. Then, ρ denoting the density of the fluid,

$$dp = \rho\, (\omega^2 y\, dy - g\, dx),$$

$$p = \rho\, (\tfrac{1}{2}\omega^2 y^2 - gx) + C.$$

At the free surface $p = 0$, and therefore

$$0 = \rho\, (\tfrac{1}{2}\omega^2 y^2 - gx) + C:$$

but this surface passes through O, hence $C = 0$, and therefore the equation to the generating curve of the free surface will be

$$2gx = \omega^2 y^2,$$

which is the equation to a parabola.

Now the volume of a paraboloid of revolution is equal to half that of the circumscribing cylinder. Hence, taking V to represent the volume of the whole fluid, and R the radius of the hollow cylinder,

$$\pi R^2 . OK + \frac{\pi\omega^2 R^4}{4g} = V \ldots\ \ldots\ldots (1).$$

Similarly, U representing the volume of the solid $aDC\beta O$,

$$\pi r^2 . OH + \frac{\pi\omega^2 r^4}{4g} = U:$$

but, since the weight of the cylinder must be equal to that of the fluid which it displaces,

$$g\rho U = W;$$

hence $$\pi r^2 . OH + \frac{\pi\omega^2 r^4}{4g} = \frac{W}{g\rho} \quad\text{............(2).}$$

From (1) and (2) we see that

$$HK + \frac{\omega^2}{4g}(R^2 - r^2) = \frac{V}{\pi R^2} - \frac{W}{\pi r^2 g\rho},$$

a result which shews that HK is less by the quantity

$$\frac{\omega^2}{4g}(R^2 - r^2),$$

when ω has any finite value than when it is equal to zero.

3. A cylindrical vessel contains a given mass of air, which revolves round its axis with a uniform angular velocity: to determine the density at any point, and the whole pressure on the curved surface of the cylinder, neglecting the effect of gravity.

Let k denote the ratio of the pressure p at any point of the air to the density ρ at the same point, ω the angular velocity, and r the distance of the point from the axis of the cylinder. Then $$dp = \rho . \omega^2 r\, dr, \quad p = k\rho,$$

and therefore $$k\frac{dp}{p} = \omega^2 r\, dr,$$

$$k \log p = C + \tfrac{1}{2}\omega^2 r^2,$$

C being an arbitrary constant.

Let Π denote the pressure at any point in the axis of the cylinder; then $$k \log \Pi = C,$$

and consequently $$k \log \frac{p}{\Pi} = \tfrac{1}{2}\omega^2 r^2,$$

$$p = \Pi \epsilon^{\frac{\omega^2 r^2}{2k}} \quad\text{..................(1),}$$

$$\rho = \frac{\Pi}{k}\epsilon^{\frac{\omega^2 r^2}{2k}} \quad\text{..................(2).}$$

Let M represent the mass of the fluid: then, h denoting the length and R the radius of the cylinder,

$$\begin{aligned} M &= \int_0^R 2\pi r\,dr\,.\,h\,.\,\rho \\ &= 2\pi h\int_0^R \rho r\,dr \\ &= \frac{2\pi h\Pi}{k}\int_0^R r\,dr\,.\,\epsilon^{\frac{\omega^2 r^2}{2k}} \\ &= \frac{2\pi h\Pi}{\omega^2}\left(\epsilon^{\frac{\omega^2 R^2}{2k}} - 1\right); \end{aligned}$$

hence, from (2), $$\rho = \frac{M\omega^2}{2\pi hk}\cdot\frac{\epsilon^{\frac{\omega^2 r^2}{2k}}}{\epsilon^{\frac{\omega^2 R^2}{2k}} - 1},$$

which determines the density at any point of the revolving air.

Also, from (1), giving Π its value, and putting $r = R$, we get for the unit of pressure at the surface of the cylinder,

$$p = \frac{M\omega^2}{2\pi h}\cdot\frac{\epsilon^{\frac{\omega^2 R^2}{2k}}}{\epsilon^{\frac{\omega^2 R^2}{2k}} - 1},$$

and therefore the whole pressure on the curved surface of the cylinder will be equal to

$$2\pi Rhp = MR\omega^2\,\frac{\epsilon^{\frac{\omega^2 R^2}{2k}}}{\epsilon^{\frac{\omega^2 R^2}{2k}} - 1}.$$

4. To find the form of equilibrium of a fluid of uniform density, acted on by an attractive force which varies directly as the distance and tends towards a point in a vertical line, about which the fluid is revolving with a uniform angular velocity.

Let x denote the altitude of any particle of the revolving fluid above a horizontal plane through the centre of force, y the distance of this particle from the axis of revolution, and r from the centre of force. Then, ω denoting the angular velocity, and μ the absolute force of attraction,

$$dp = \rho\,(\omega^2 y\,dy - \mu r\,dr - g\,dx).$$

Integrating we have, c^2 denoting a constant quantity,

$$\frac{p}{\rho} = \tfrac{1}{2}c^2 + \tfrac{1}{2}\omega^2 y^2 - \tfrac{1}{2}\mu r^2 - gx$$
$$= \tfrac{1}{2}c^2 - \tfrac{1}{2}(\mu - \omega^2)\, y^2 - \tfrac{1}{2}\mu x^2 - gx,$$

whence, putting $p = 0$, we obtain for the equation to the generating curve of the free surface,

$$c^2 = (\mu - \omega^2)\, y^2 + \mu x^2 + 2gx,$$

which is the equation to a conic section.

5. A paraboloid of revolution, the axis of which is vertical, contains a given quantity of incompressible fluid; the fluid is revolving with a given angular velocity about the axis of the paraboloid: to determine the normal pressure on the whole of the concave surface of the paraboloid which is in contact with the fluid.

Let y be the distance of any point of the revolving fluid from the axis of the paraboloid, x its distance from a horizontal plane touching the paraboloid at its vertex, then for the equilibrium we have

$$dp = \rho\,(\omega^2 y dy - g dx);$$

integrating and adding an arbitrary constant, we get

$$p = \rho\,(\tfrac{1}{2}\omega^2 y^2 - gx) + C.$$

Let a represent the value of x at the point of the free surface in which it is intersected by the axis of revolution; then, putting $x = a$, $y = 0$, $p = 0$, we see that

$$0 = -g\rho a + C;$$

and consequently

$$p = \rho\,\{\tfrac{1}{2}\omega^2 y^2 - g\,(x - a)\}.$$

Putting $p = 0$, we obtain for the equation to the generating curve of the free surface

$$\omega^2 y^2 = 2g\,(x - a)\,\ldots\ldots\ldots\ldots\ldots(1),$$

which is the equation to a parabola.

Let the equation to the generating parabola of the vessel be

$$y^2 = lx\,\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

At the intersection of (1) and (2), x' and y' denoting the values of x and y,

$$\omega^2 l x' = 2g\,(x' - a),$$

$$x' = \frac{2ga}{2g - \omega^2 l}, \qquad y'^2 = \frac{2gla}{2g - \omega^2 l}.$$

Now the volume of a paraboloid of revolution is equal to half that of the circumscribing cylinder; hence, the volume of the fluid being V, which will be equal to the difference of the volumes of two paraboloids,

$$V = \tfrac{1}{2}\pi y'^2 \{x' - (x' - a)\}$$

$$= \frac{\pi g l a^2}{2g - \omega^2 l},$$

$$a^2 = \frac{2g - \omega^2 l}{\pi g l} \cdot V,$$

which determines the value of a.

The whole pressure of the fluid on the vessel will be equal to

$$2\pi \int_0^{x'} y\,ds\,.\,p = 2\pi\rho \int_0^{x'} y ds\,.\,\{\tfrac{1}{2}\omega^2 y^2 - g\,(x - a)\}:$$

but, since $y^2 = lx$, it is easily seen that

$$yds = l^{\frac{1}{2}}(x + \tfrac{1}{4}l)^{\frac{1}{2}}dx;$$

hence the required pressure is equal to

$$2\pi\rho l^{\frac{1}{2}} \int_0^{x'} dx\,(x + \tfrac{1}{4}l)^{\frac{1}{2}}\{ga - (g - \tfrac{1}{2}\omega^2 l)\,x\}.$$

Performing the integration and simplifying, we shall obtain for the final result the following expression for the whole pressure, viz.

$$\tfrac{1}{15}\pi\rho l^{\frac{1}{2}}(x' + \tfrac{1}{4}l)^{\frac{3}{2}}\,.\,(8ga + 2gl - \omega^2 l^2)$$
$$- \tfrac{1}{120}\pi\rho l^2\,(20ga + 2gl - \omega^2 l^2).$$

6. A hemispherical bowl, filled with fluid of a given density, is placed in an inverted position on a horizontal plane, and the whole system revolves round a vertical axis passing through the centre of the sphere with a given angular velocity: to determine the weight of the bowl that it may just prevent the fluid from escaping.

Let the axis of the bowl be taken as the axis of x, and any radius of its rim as the axis of y. Let ω represent the angular velocity of the fluid. Then,

$$dp = \rho\,(\omega^2 y dy - g dx),$$
$$p = \rho\,(\tfrac{1}{2}\omega^2 y^2 - gx) + C,$$

C being an arbitrary constant.

Supposing for a moment the bowl not to be full of fluid, we shall have for the equation to the free surface

$$0 = \rho\,(\tfrac{1}{2}\omega^2 y^2 - gx) + C.$$

Let c represent the distance of the intersection of the free surface with the axis of x from the origin of coordinates: then

$$0 = -g\rho c + C,$$

and therefore the unit of pressure at any point is given by the equation

$$p = \rho\,\{\tfrac{1}{2}\omega^2 y^2 + g\,(c - x)\}\ldots\ldots\ldots\ldots(1),$$

and the equation to the free surface will be

$$\tfrac{1}{2}\omega^2 y^2 = g\,(x - c)\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Suppose now the paraboloidal cavity represented by the equation (2) to be indefinitely small, which is the same thing as supposing the vessel to be just full of fluid: then, r being the radius of the interior of the bowl, $c = r$, and the equation (1) is reduced to

$$p = \rho\,\{\tfrac{1}{2}\omega^2 y^2 + g\,(r - x)\}\ldots\ldots\ldots\ldots\ldots(3).$$

The whole vertical pressure of the fluid on the vessel will be equal to

$$\int 2\pi y ds\,.\,\frac{x}{r}\,.\,p,$$

or, since yds is equal to rdx, will be equal to

$$\begin{aligned}
&2\pi \int p x dx \\
&= 2\pi\rho \int_0^r x dx\,(\tfrac{1}{2}\omega^2 r^2 + gr - gx - \tfrac{1}{2}\omega^2 x^2), \quad \text{by (3)}, \\
&= 2\pi\rho\,(\tfrac{1}{4}\omega^2 r^4 + \tfrac{1}{2}gr^3 - \tfrac{1}{3}gr^3 - \tfrac{1}{8}\omega^2 r^4) \\
&= \tfrac{1}{12}\pi\rho r^3\,(3\,\omega^2 r + 4g).
\end{aligned}$$

Let W represent the weight of the vessel: then, since this by the nature of the problem must be equal to the total vertical pressure which the fluid exerts upon it, we have

$$W = \tfrac{1}{12}\pi\rho r^3\,(3\omega^2 r + 4g),$$

which determines W.

7. A small tube is so bent as to form three sides of a square. Being placed with its extreme branches vertical and half filled with fluid, it is made to revolve about a vertical axis passing through a given point in the horizontal branch: to compare the heights of the two surfaces of the fluid.

Let x represent the distance of any particle of the fluid from the axis of revolution, and y the altitude of the particle above the horizontal branch of the tube: then, ω denoting the angular velocity, and p the pressure at this particle,

$$dp = \rho\,(\omega^2 x dx - g\,dy),$$

$$p = \rho\,(\tfrac{1}{2}\omega^2 x^2 - gy) + C,$$

C being an arbitrary constant.

Let h, k, denote the altitudes, above the horizontal branch of the tube, of the surfaces of the fluid in the two vertical branches, and a, b, the distances of the axis of revolution from the two vertical branches. Then

$$0 = \rho\,(\tfrac{1}{2}\omega^2 a^2 - gh) + C,$$

$$0 = \rho\,(\tfrac{1}{2}\omega^2 b^2 - gk) + C,$$

and therefore

$$h - k = \frac{\omega^2}{2g}(a^2 - b^2)\ldots\ldots\ldots\ldots(1).$$

But, the tube being half full of fluid, there is

$$h + k = \tfrac{1}{2}(a + b)\ldots\ldots\ldots\ldots(2).$$

From (1) and (2) we see that

$$h = \frac{a+b}{4g}\{g + \omega^2(a - b)\},\quad k = \frac{a+b}{4g}\{g - \omega^2(a - b)\},$$

and therefore
$$\frac{h}{k} = \frac{g + \omega^2(a - b)}{g - \omega^2(a - b)}.$$

8. Two fluids, which do not mix, are placed in a cylinder the axis of which is vertical, and the system is then made to revolve with a given angular velocity about its axis: to determine the equation to the common surface of the two fluids, the radius of the cylinder and the volume of the lower fluid being known.

Let the axis of the cylinder be taken as the axis of x, and

a radius of its base as the axis of y. Then we have, for the equilibrium of the upper fluid, ρ denoting its density,

$$dp = \rho\,(\omega^2 y dy - g dx),$$

$$p = \rho\,(\tfrac{1}{2}\omega^2 y^2 - gx) + C \ldots\ldots\ldots (1).$$

Also, for the equilibrium of the lower fluid, ρ' denoting its density,

$$p = \rho'\,(\tfrac{1}{2}\omega^2 y^2 - gx) + C' \ldots\ldots\ldots (2).$$

At the common surface of the two fluids, the values of p in the equations (1) and (2) must be the same: hence the equation to the common surface of the two fluids will be

$$(\rho - \rho')\,(\tfrac{1}{2}\omega^2 y^2 - gx) = C' - C \ldots\ldots\ldots (3).$$

Let a denote the distance of the intersection of the surface (3) and the axis of x from the origin: then

$$-(\rho - \rho') \,.\, ga = C' - C,$$

and therefore (3) becomes

$$\tfrac{1}{2}\omega^2 y^2 = g\,(x - a) \ldots\ldots\ldots\ldots (4),$$

which shews that the common surface of the two fluids is a paraboloid of revolution.

Let u denote the volume of the lower fluid: then, since the volume of a paraboloid of revolution is equal to half that of the circumscribing cylinder,

$$u = \pi r^2 x - \tfrac{1}{2}\pi r^2\,(x - a) = \tfrac{1}{2}\pi r^2\,(x + a),$$

the value of x being derived from the equation (4) when r, the radius of the cylinder, is substituted for y. Hence

$$u = \tfrac{1}{2}\pi r^2 \left(\frac{\omega^2 r^2}{2g} + 2a\right),$$

$$a = \frac{u}{\pi r^2} - \frac{\omega^2 r^2}{4g},$$

and therefore the equation (4) becomes

$$\tfrac{1}{2}\omega^2 y^2 = gx - \frac{gu}{\pi r^2} + \tfrac{1}{4}\omega^2 r^2.$$

9. A hemisphere with its axis vertical contains a quantity of fluid, the volume of which is equal to a quarter of that of the hemisphere; the fluid is revolving with a uniform angular velocity and just reaches the rim of the hemisphere: to determine the angular velocity of the fluid.

Let centre of the sphere be taken as the origin of coordinates, the axis of x being the axis of the hemisphere and that of y a radius of the rim. We shall have, for the pressure at any point of the revolving fluid,

$$dp = \rho\,(\omega^2 y\,dy + g\,dx),$$

$$p = \rho\,(\tfrac{1}{2}\omega^2 y^2 + gx) + C,$$

C being an arbitrary constant. Let c denote the distance of the lowest point of the free surface from the origin of coordinates: then

$$0 = \rho gc + C:$$

$$p = \rho\,\{\tfrac{1}{2}\omega^2 y^2 + g\,(x - c)\}.$$

At the free surface therefore

$$y^2 = \frac{2g}{\omega^2}(c - x).$$

The volume of the cavity of the revolving fluid is equal to

$$\tfrac{1}{2}\pi a^2 . c = \tfrac{1}{2}\pi a^2 . \frac{\omega^2 a^2}{2g} = \frac{\pi\omega^2 a^4}{4g},$$

a being the radius of the sphere.

The volume of the hemisphere is equal to

$$\tfrac{2}{3}\pi a^3.$$

Hence, by the condition of the problem,

$$\frac{\pi\omega^2 a^4}{4g} = \tfrac{3}{4} . \tfrac{2}{3}\pi a^3, \quad \omega^2 = \frac{2g}{a}.$$

10. A slender horizontal tube, filled with inelastic fluid, is placed with one extremity coinciding with an attractive centre of force, the intensity of which varies as the m^{th} power of the distance, about which it revolves with an angular velocity ω: to determine the position of that point of the tube at which the fluid pressure is the same as at the fixed end.

If μ denote the absolute force of attraction, and r the distance of any point of the tube from the centre of force,

$$dp = \rho\,\{\omega^2 r dr - \mu r^m dr\},$$

$$p = \Pi + \rho\left\{\tfrac{1}{2}\omega^2 r^2 - \frac{\mu}{m+1}\,r^{m+1}\right\},$$

Π denoting the pressure at the fixed extremity of the tube.

If $p = \Pi$: then
$$\tfrac{1}{2}\omega^2 r^2 = \frac{\mu}{m+1} r^{m+1},$$
$$r = \left\{\frac{\omega^2 (m+1)}{2\mu}\right\}^{\frac{1}{m-1}},$$
which determines the required distance.

11. A cylindrical vessel, the axis of which is vertical, is furnished with a lid which turns round a hinge in its circumference. Supposing it to be just filled with fluid, and made to revolve with a given angular velocity round its axis, to find what the weight of the lid must be in order that no fluid may escape.

If x represent the depth of any particle of the fluid below the lid, and r its distance from the axis of the cylinder, then
$$dp = \rho\,(\omega^2 r dr + g dx),$$
$$p = \rho\,(\tfrac{1}{2}\omega^2 r^2 + gx + C).$$

Now, supposing the cylinder to be indefinitely nearly full, there will be an indefinitely small free paraboloidal surface, and therefore, for indefinitely small simultaneous values of x and y, p will be equal to zero: hence, ultimately, when the vessel is just full, $C = 0$; and therefore
$$p = \rho\,(\tfrac{1}{2}\omega^2 r^2 + gx).$$
At any point of the lid, $x = 0$, and therefore
$$p = \tfrac{1}{2}\rho\omega^2 r.$$

Let θ denote the inclination of r, at any point of the lid, to the line joining the hinge and the centre of the lid: then, a denoting the radius of the cylinder, the moment of the whole pressure of the fluid to turn the lid about the hinge will be equal to
$$\int_0^a \int_0^{2\pi} \tfrac{1}{2}\rho\omega^2 r^2 \,.\, r d\theta dr \,.\, (a + r\cos\theta)$$
$$= \tfrac{1}{2}\rho\omega^2 a^5 \int_0^{2\pi} (\tfrac{1}{4} + \tfrac{1}{5}\cos\theta)\, d\theta$$
$$= \tfrac{1}{4}\pi\rho\omega^2 a^5.$$

Let W denote the weight of the lid: then, under the conditions of the problem,
$$Wa = \tfrac{1}{4}\pi\rho\omega^2 a^5,$$
$$W = \tfrac{1}{4}\pi\rho\omega^2 a^4.$$

12. A cylindrical vessel partly filled with fluid is attached to a weight by a string which passes over a fixed pully: supposing the fluid to revolve with a given angular velocity as the vessel is ascending or descending, to find the form of the free surface.

Let M denote the sum of the masses of the vessel and fluid, and P the mass of the weight. Then the acceleration f of the vessel and fluid upwards, supposing P to be greater than M, will be equal to

$$\frac{P-M}{P+M}g.$$

Now the fluid is acted on at every point by the force of gravity and the centrifugal force arising from its rotation, and, at its lower surface, by the pressure of the vessel. But the relative state of the fluid and vessel will not be affected if we impress upon every molecule of the system $M+P$, the same accelerating force exactly opposite to its motion. Suppose then f to be so impressed upon the system: the vessel being thus deprived of acceleration, we may treat it as if it were at rest, the fluid which it contains being on the present hypothesis acted on at each point by g downwards, f downwards, and the centrifugal force.

Now $$g+f=g+\frac{P-M}{P+M}g=\frac{2Pg}{P+M}.$$

Hence, for the equilibrium of the fluid, there is

$$dp=\rho\left(\omega^2 y dy-\frac{2Pg}{P+M}dx\right),$$

y denoting the distance of any molecule from the axis of revolution, and x its altitude above an assigned point in this axis. Integrating, we have

$$p=\rho\left(\tfrac{1}{2}\omega^2 y^2-\frac{2Pgx}{P+M}+C\right),$$

C being a constant.

At the free surface

$$0=\tfrac{1}{2}\omega^2 y^2-\frac{2Pgx}{P+M}+C.$$

Let the point where the surface of the fluid is intersected by

the axis of revolution be taken as the origin of coordinates: then we have

$$y^2 = \frac{4Pg}{\omega^2 (P+M)} x,$$

which shews that the surface is a paraboloid of revolution, the latus rectum of which is equal to

$$\frac{4Pg}{\omega^2 (P+M)}.$$

This result will not be affected if M be greater than P, and the vessel accordingly be descending instead of ascending.

13. A fluid mass revolves about a fixed line, and its particles are attracted towards a point in this line with a constant force: to determine the form of a surface of equal pressure within the fluid, the angular velocity, the constant force, and the radius of the section of this surface made by a plane through the centre of force at right angles to the axis of revolution, being all supposed to be known.

If ω = the angular velocity, f = the attractive force, c = the given radius of the section; then, r denoting the distance of any point of the required surface from the centre of force and y from the axis of revolution, the equation to the required surface of equal pressure will be

$$\omega^2 (y^2 - c^2) = 2f(r - c).$$

If b be the value of c at the free surface, the equation to the free surface will be

$$\omega^2 (y^2 - b^2) = 2f(r - b).$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 229.

14. To determine the form of the surface of equal pressure and of the free surface, the conditions being the same as in the preceding problem, excepting that the central force, instead of being constant, varies directly as the distance.

If μ represent the absolute force of attraction, the equations to the free surface and to the surface of equal pressure will be

respectively, if we retain the notation of the preceding problem,

$$\omega^2 (y^2 - b^2) = \mu (r^2 - b^2),$$

and

$$\omega^2 (y^2 - c^2) = \mu (r^2 - c^2).$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 230.

15. An inverted cone is filled with fluid, and made to revolve round its axis, which is vertical: to find the quantity spilt as the angular velocity increases from 0 to ∞.

If b denote the radius of the base of the cone, and h its altitude, then, for an angular velocity ω, the quantity spilt will be equal to

$$\frac{\pi \omega^2 b^4}{4g},$$

and the cone will be entirely empty, provided that

$$\omega^2 = \frac{4gh}{3b^2}.$$

16. To determine the greatest angular velocity with which a sphere filled with fluid may be whirled round its vertical diameter, so as to allow the whole of the fluid to escape by a small orifice at its lowest point.

If r denote the radius of the sphere, g the force of gravity, and ω the required angular velocity,

$$\omega^2 = \frac{g}{r}.$$

17. A hemispherical vessel contains fluid which is revolving about its axis, which is vertical, with such an angular velocity as to bring the fluid just up to the brim, the volume of the fluid being equal to half that of the vessel: to determine the whole pressure on the surface of the vessel.

If ρ denote the density of the fluid and a the radius of the vessel, the required pressure will be equal to

$$\tfrac{1}{3}\pi g \rho a^3.$$

18. A cylindrical vessel of given dimensions containing air, revolves about its axis with a given angular velocity: to deter-

mine the law of the variation of the pressure from the axis to the surface of the cylinder.

If p denote the pressure at the axis, and p' at any point at a distance r from the axis; then, ω representing the angular velocity and k a constant quantity,

$$p' = p \cdot \varepsilon^{\frac{\omega^2 r^2}{2k}}.$$

19. If each particle of a mass of fluid be attracted towards a centre of force varying as the distance, and also by a given constant force acting in parallel lines; to find the form of equilibrium of the fluid if a motion of rotation be given to it about an axis passing through the centre of force and parallel to the direction of the constant force.

Let the centre of force be taken as the origin of coordinates, and the axis of revolution as the axis of x; then, μ denoting the absolute force of attraction, f the constant force, and c a constant quantity dependent upon the volume of the fluid,

$$c^2 = \mu x^2 + (\mu - \omega^2) y^2 - 2fx,$$

will be the equation to a plane section of the free surface made by a plane through the axis of revolution.

20. A hollow vertical prism, just filled with heavy incompressible fluid, revolves round one edge with a given angular velocity: to find the centre of pressure on the base, which is an isosceles right-angled triangle with its right-angle at the axis of revolution.

If h = the altitude of the prism, a = either of the equal sides of the base, ω = the angular velocity, and $\bar{x}$ = the distance of the centre of pressure from the axis of revolution,

$$\bar{x} = \tfrac{2}{5} a \cdot \frac{\omega^2 a^2 + 10gh}{\omega^2 a^2 + 12gh}.$$

21. A cube, filled with fluid and closed, revolves uniformly about one of its edges placed in a vertical position: to find the pressure upon each face of the cube, the pressure at the highest point of the axis of revolution being supposed to be zero, and to

determine the tension of a string which connects the adjacent edges of two vertical faces which sustain different pressures, and which alone prevents their revolving about their other vertical edges.

Let ρ = the density of the fluid, ω = the angular velocity, a = the length of each edge of the cube; then

Pressure on the lower of the horizontal faces $= \frac{1}{3}\rho a^3(a\omega^2 + 3g)$;
Pressure on the higher of the horizontal faces $= \frac{1}{3}\rho\omega^2 a^4$;
Pressure on either of the vertical faces adjacent to the axis of revolution $= \frac{1}{6}\rho a^3(\omega^2 a + 3g)$;
Pressure on either of the vertical faces remote from the axis of revolution $= \frac{1}{6}\rho a^3(4\omega^2 a + 3g)$; and
Tension of the string $= \frac{1}{24}\rho a^3(58\,\omega^4 a^2 + 120\,\omega^2 ag + 72g^2)^{\frac{1}{2}}$.

HYDRODYNAMICS.

CHAPTER I.

EFFLUX OF FLUIDS FROM VESSELS.

SECTION I.

Small Orifices. Incompressible Fluids acted on by Gravity.

WHEN fluid issues from a vessel in which it is contained through a small orifice, the surface of the fluid being supposed to be subject to the action of the atmosphere into which the efflux takes place, the stream keeps contracting for some distance after escaping from the orifice, and afterwards acquires a permanent cylindrical form. This thin uniform cylinder of fluid is called the *vena contracta*. The section of the vena contracta may be considered as about $\frac{5}{8}$ths of the actual orifice.*

Let v represent the velocity and k the area of a section of the vena contracta, K the area and V the velocity of the surface of the descending fluid, at any time t. Then, x denoting the altitude of the surface of the fluid at the time t above the orifice, we have for the determination of the motion of the fluid,

$$v = \left\{\frac{2gx}{1 - \frac{k^2}{K^2}}\right\}^{\frac{1}{2}} = (2gx)^{\frac{1}{2}}, \text{ approximately,}$$

$\frac{k}{K}$ being a very small quantity; and

$$VK = vk,$$

or

$$kvdt = - Kdx.$$

The truth of the equation $VK = vk$ depends upon the supposition that the quantity of the fluid in the vessel is not

* Bossut: *Traité d'Hydrodynamique.* Rennie's *Report to the British Association*, 1834.

replenished by the influx of fresh fluid. Suppose, however, that fluid enters into the vessel with a velocity v' through a small aperture k': then, instead of the equation $VK = vk$, we shall have

$$vk - v'k' = VK,$$

or

$$(vk - v'k')\, dt = -\, Kdx.$$

The equation $v = (2gx)^{\frac{1}{2}}$ expresses the fact that the velocity of the issuing fluid is that due to a body falling freely in vacuum through a vertical space equal to the depth of the orifice below the surface of the fluid. This law of efflux was ascertained experimentally by Torricelli, by whom it was announced as the result of observation at the end of a short Treatise, *De motu gravium naturaliter accelerato*, printed in 1643. Theoretical explanations of this law, only partially satisfactory, were first supplied by Newton,* in the second edition of his *Principia*, which appeared in the year 1714; and by Varignon,† in a memoir presented to the Academy of Sciences of Paris, in the year 1694. The first satisfactory demonstration of Torricelli's law was given by Daniel Bernoulli, in the Memoirs of the Academy of St. Petersburgh for the year 1726, by the application of the principle of the conservation of vis viva, which forms the basis of his great work, entitled *Hydrodynamica seu de viribus et motibus fluidorum commentarii*, printed in the year 1738. The principle itself of the conservation of vis viva had not at that time, however, been established generally: the demonstration of Daniel Bernoulli, although unobjectionable in the present state of science, accordingly failed to satisfy the minds of the philosophers of his day. Attempts were made by Maclaurin and John Bernoulli to obviate the uncertainty at that time attaching itself to Daniel Bernoulli's explanation by deducing the law of efflux from the fundamental principles of mechanics: Maclaurin's demonstration was published in his *Treatise of Fluxions* (Book I. Chap. 12, Art. 137,

* In the first edition of the *Principia*, published in the year 1687, Newton had concluded the velocity of efflux to be only half the velocity due to the depth of the fluid in the vessel: this error arose from his taking the actual orifice instead of the vena contracta; an inaccuracy which he corrected in the second edition.

† See *Mém. Acad. Paris*, ann. 1703, p. 245.

&c.), and that of John Bernoulli, who appears, from a letter addressed to him by Euler, to have been in possession of the principal part of his theory as early as the year 1726, in his *Hydraulica*, printed after his death, at the end of his Works, in the year 1732. The obscurity, however, and want of rigour in these researches, especially in that by Maclaurin, who appears to have pre-arranged his process in accommodation to the results at which he wished to arrive, rendered them of little value. The fulfilment of the object which these two mathematicians had proposed to themselves was reserved for D' Alembert, who, assuming, after Daniel Bernoulli, the hypothesis of *parallel sections*, deduced the theory of the motion of fluids in vessels from his general principle of Dynamics, at the end of his *Traité de Dynamique*, published in the year 1743, and the year following, with numerous details, in his *Traité des Fluides*. For additional information on the efflux of fluids, the reader is referred also to a memoir by Borda in the *Mémoires de l'Académie des Sciences de Paris*, for 1766, p. 579.

A peculiar method of expressing the conditions of the motion of fluids, under the most general circumstances, by means of equations, without having recourse to any particular hypothesis, was discovered by D'Alembert, by the application of his principle of dynamics, in his *Essai d'une nouvelle Théorie sur la resistance des Fluides*, published in 1752, on the occasion of a prize, proposed by the Academy of Berlin, on the Resistance of Fluids. It is to Euler, however, advancing beyond the first great step taken by D'Alembert, that we are indebted for the most general and most simple analytical expression of the general motion of fluids, in the form given by Poisson in his *Traité de Mécanique*, tom. II., *Livre Sixième*, and other modern treatises. His researches on this subject were published in the *Mémoires de l'Académie des Sciences de Berlin*, for the year 1755. The fundamental equations of hydrodynamics discovered by Euler were afterwards demonstrated by Lagrange, by the application of a most refined and elegant analysis, in the second volume of his *Traité de Mécanique*, based upon the assumption that a fluid consists of an indefinitely great number of small molecules freely moveable among each other.

1. A vessel, formed by the revolution of a semi-cubical parabola about its axis, which is vertical, is filled with fluid till the diameter of the surface is equal to twice its distance from the apex: to find the time in which the fluid will be discharged through a small hole in the apex.

The equation to the semi-cubical parabola is

$$ay^2 = x^3.$$

Hence
$$\frac{dx}{dt} = -\frac{kv}{K} = -\frac{k\,(2gx)^{\frac{1}{2}}}{\pi y^2} = -\frac{ka}{\pi}\cdot\frac{(2gx)^{\frac{1}{2}}}{x^3},$$

$$dt = -\frac{\pi}{ka\,(2g)^{\frac{1}{2}}}\cdot x^{\frac{5}{2}}dx,$$

and therefore, the limits of x being a, 0, the required time will be equal to

$$-\frac{\pi}{ka\,(2g)^{\frac{1}{2}}}\cdot\int_a^0 x^{\frac{5}{2}}dx$$

$$=\frac{\pi}{ka\,(2g)^{\frac{1}{2}}}\cdot\tfrac{2}{7}a^{\frac{7}{2}} = \frac{2^{\frac{1}{2}}\pi a^{\frac{5}{2}}}{7kg^{\frac{1}{2}}}.$$

2. To find the time in which an ellipsoid will empty itself through a given small aperture at the extremity of one of its principal axes, which is vertical.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the axis of z extending vertically downwards.

Then, for the determination of the velocity of the descending surface, we have

$$\frac{dz}{dt} = \frac{k\,\{2g\,(c-z)\}^{\frac{1}{2}}}{K};$$

but, writing the equation to the ellipsoid under the form

$$\frac{x^2}{a^2\left(1-\dfrac{z^2}{c^2}\right)} + \frac{y^2}{b^2\left(1-\dfrac{z^2}{c^2}\right)} = 1,$$

we see that $$K = \pi ab\left(1 - \frac{z^2}{c^2}\right):$$

hence $$dt = \frac{\pi ab}{k\,(2g)^{\frac{1}{2}}} \cdot \frac{1 - \frac{z^2}{c^2}}{(c-z)^{\frac{1}{2}}} \cdot dz$$

$$= \frac{\pi ab}{kc^2 (2g)^{\frac{1}{2}}} \cdot (c+z)\,(c-z)^{\frac{1}{2}} \cdot dz,$$

and the required time is given by the equation

$$t = \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}} \cdot \int_{-c}^{+c} \{2c - (c-z)\}\,(c-z)^{\frac{1}{2}} \cdot dz$$

$$= \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}} \cdot \Big[_{-c}^{+c}\left\{-\tfrac{4}{3}c\,(c-z)^{\frac{3}{2}} + \tfrac{2}{5}(c-z)^{\frac{5}{2}}\right\}$$

$$= \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}} \cdot \left\{\tfrac{4}{3}c \cdot 2^{\frac{3}{2}} \cdot c^{\frac{3}{2}} - \tfrac{2}{5} \cdot 2^{\frac{5}{2}} \cdot c^{\frac{5}{2}}\right\}$$

$$= \frac{\pi abc^{\frac{1}{2}}}{k\,(2g)^{\frac{1}{2}}} \cdot \left(\tfrac{1}{3} \cdot 2^{\frac{7}{2}} - \tfrac{1}{5} \cdot 2^{\frac{7}{2}}\right)$$

$$= \frac{16\pi ab}{15k} \cdot \left(\frac{c}{g}\right)^{\frac{1}{2}}.$$

3. A vertical cylindrical vessel, filled with a fluid, begins to empty itself through a small orifice in its base: the vessel is constantly kept full by another fluid entering from above, which is kept separate from the lower fluid by a very thin disk: to find the time of the efflux of the lower fluid.

Let h represent the altitude of the cylinder, x the depth of the disk below the rim of the vessel after a time t from the commencement of the efflux, ρ the density of the upper, and σ of the lower fluid, and A the area of a horizontal section of the cylinder. Then, the velocity of the efflux at the time t being the same as if the orifice were at the bottom of a column of the lower fluid of an altitude equal to

$$h - x + \frac{\rho}{\sigma}x = h + \frac{\rho - \sigma}{\sigma}\,x,$$

we have
$$A\frac{dx}{dt}=k(2g)^{\frac{1}{2}}\left(h+\frac{\rho-\sigma}{\sigma}x\right)^{\frac{1}{2}},$$
$$dt=\frac{A}{k(2g)^{\frac{1}{2}}}\cdot\frac{dx}{\left(h+\frac{\rho-\sigma}{\sigma}x\right)^{\frac{1}{2}}},$$

whence, integrating from $x=0$ to $x=h$, we see that the required time is equal to

$$\frac{2A}{k(2g)^{\frac{1}{2}}}\cdot\frac{\sigma}{\rho-\sigma}\cdot{}_{0}^{h}\left\{h+\frac{\rho-\sigma}{\sigma}x\right\}^{\frac{1}{2}}$$
$$=\frac{2Ah^{\frac{1}{2}}}{k(2g)^{\frac{1}{2}}}\cdot\frac{\sigma^{\frac{1}{2}}}{\rho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}}.$$

4. A vertical prismatic vessel empties itself in a certain time through a small orifice in its base: to compare the volume of the fluid discharged with the volume which would have been discharged in the same time, had the surface of the fluid been constantly maintained at its original altitude.

The quantity discharged in the latter case would be double of that discharged in the former.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 261.

5. To find the time in which a hemisphere, filled with fluid, will empty itself through a small orifice in its vertex, the axis of the hemisphere being vertical.

If r denote the radius of the sphere, the time required will be equal to

$$\frac{14\pi r^{\frac{3}{2}}}{15(2g)^{\frac{1}{2}}k}.$$

Encycl. Metrop. Mix. Sc. vol. I. p. 204.

6. To find the time in which a hemisphere, filled with fluid, will empty itself through a small orifice in its base, its axis being vertical.

If r denote the radius of the sphere, the time of efflux will be equal to

$$\frac{8\pi r^{\frac{3}{2}}}{5(2g)^{\frac{1}{2}}k}.$$

Encycl. Metrop. Mix. Sc., vol. I. p. 204.

7. To find the time in which a paraboloid of revolution, full of fluid, will empty itself through a small orifice in its vertex, the axis of the paraboloid being vertical.

If h denote the altitude of the vessel and l the latus rectum, the time of efflux will be equal to

$$\frac{2\pi l h^{\frac{3}{2}}}{3\,(2g)^{\frac{1}{2}}\,k}.$$

Encycl. Metrop. Mix. Sc., vol I. p. 205.

8. To find the time in which the fluid, in a vertical prismatic vessel, will discharge itself through a small orifice in its base.

If a denote the initial depth of the fluid, and A the area of a horizontal section of the prism, the time required will be equal to

$$\frac{2Aa^{\frac{1}{2}}}{k(2g)^{\frac{1}{2}}}.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 259, 260.
Encycl. Metrop. Mix. Sc., vol. I. p. 204.

9. To find the time in which the surface of the fluid in a conical vessel will subside to half its original altitude above the vertex through a small orifice in the vertex, the axis of the cone being vertical.

If h denote the height of the cone and r the radius of its base, the required time will be equal to

$$\frac{\pi r^2 h^{\frac{1}{2}}(2^{\frac{5}{2}} - 1)}{20kg^{\frac{1}{2}}}.$$

10. Fluid issues from innumerable holes in the surface of a vertical cylinder kept constantly full: to determine the form of the bounding surface of the issuing fluid.

The bounding surface will be a truncated cone, its narrow end being the upper end of the cylinder; and the vertex of the entire cone will be in the axis of the cylinder produced upwards, at a distance from the upper end of the cylinder equal to its radius.

11. To find the time in which a vessel, formed by the revolution of a cycloid about its axis, which is vertical, will empty

itself, after being filled with fluid, through a small orifice at its vertex.

If a denote the radius of the generating circle, the time of efflux will be equal to

$$\frac{\pi a^{\frac{5}{2}}}{k g^{\frac{1}{2}}}\left(2\pi^2 - \frac{8^3}{45}\right).$$

Moseley's *Hydrostatics*, p. 154.

12. To find the time in which the fluid contained in a vessel, formed by the revolution of the curve $y^4 = a^3 x$ about the axis of x, will descend through any proposed space, the axis of the vessel being vertical, supposing a small orifice to be made at the vertex.

If s denote the space through which the fluid sinks in a time t,

$$t = \frac{\pi a^{\frac{3}{2}} s}{(2g)^{\frac{1}{2}} k},$$

which shews that equal spaces are described in equal times.

John Bernoulli: *Hydraulicæ Pars* 2, *Opera*, tom. IV. p. 481.
Mariotte: *Mouv. des Eaux*, Part. III. disc. 3.
Varignon: *Mem. Acad. Par.* 1694, 1699.
Bossut: *Traité d' Hydrodynamique*, tom. I. p. 255.

SECTION II.

Small Orifices. General Conditions.

In the preceding section we have considered the problem of the efflux of fluids through small orifices, supposing gravity to be the only acting force and the fluid to issue into the same atmosphere with which its surface is in contact. In this section we shall consider the problem under a more general point of view.

Let p denote the pressure at any point in the fluid, u the velocity of the fluid at this point, $X, Y, Z,$ the components of the accelerating force on the molecule of fluid parallel to three

fixed rectangular axes. Then, by the equation of steady motion, which, the orifice being very small, may be here applied without sensible error,

$$p = \rho\int(Xdx + Ydy + Zdz) - \tfrac{1}{2}\rho u^2 + C,$$

C denoting the constant introduced by the integration.

If p', p'', denote the units of pressure to which the surface of the fluid and the issuing fluid are respectively subject; then, u being of inconsiderable magnitude at the surface, which is supposed to be nearly of the form of equilibrium during the whole motion,

$$p' = \rho\int(Xdx + Ydy + Zdz) + C \ldots\ldots\ldots (1),$$

x, y, z, in the integral representing the coordinates of any point in the surface of the fluid; and

$$p'' = \rho\int(Xdx + Ydy + Zdz) - \tfrac{1}{2}\rho v^2 + C \ldots\ldots (2),$$

x, y, z, denoting the coordinates of the orifice, and v the velocity of efflux.

The equation (2) gives v in terms of C; suppose accordingly that, $\phi(C)$ denoting a function of C,

$$v = \phi(C).$$

Now, if A denote the volume of the fluid in the vessel at any time t, it is plain that

$$kvdt = -dA;$$

and, from the equation (1) together with the equation to the vessel, we may obtain A in terms of C: let $A = f(C)$, a function of C. Hence, for the determination of the time of efflux of any portion of the fluid,

$$dt = -\frac{f'(C)\,dC}{k\phi(C)},$$

$$t = -\frac{1}{k}\int_{C_1}^{C_2}\frac{f'(C)}{\phi(C)}\,dC,$$

C_1 denoting the value of C, when A has its initial value, and C_2 the value of C when A has the value corresponding to the state of the fluid at the end of the time t.

If the fluid be revolving about a fixed line, we may consider it, as far as the motion of rotation is concerned, as being at rest

in regard to three axes fixed in the fluid, provided that the centrifugal force be taken into consideration together with the other forces to which the fluid is subject, the accelerating forces being supposed to act either parallel to the axis of revolution or towards fixed points within it.

1. The fluid in an open vessel, in the form of a paraboloid of revolution, revolves round its axis, which is vertical, with such an angular velocity that it just reaches the rim: the volume of the fluid is such that, if it were at rest, its surface would bisect the axis of the paraboloid: to compare the time in which the vessel will empty itself into the atmosphere through a small orifice at the vertex with the time which would be required if the fluid were not rotating.

Let x represent the altitude of any molecule of the fluid above the tangent plane at the vertex, and y its distance from the axis of the cylinder. Then, by the equation of steady motion,

$$\begin{aligned} p &= \rho\int(\omega^2 y\,dy - g\,dx) - \tfrac{1}{2}\rho u^2 \\ &= \rho\left(\tfrac{1}{2}\omega^2 y^2 - gx\right) - \tfrac{1}{2}\rho u^2 + C \;\ldots\ldots\ldots\; (1), \end{aligned}$$

u being the component of the velocity of the fluid molecule which does not depend upon the rotation of the fluid.

At the surface of the fluid u may evidently be regarded as a very small quantity, and therefore, approximately,

$$p = \rho\left(\tfrac{1}{2}\omega^2 y^2 - gx\right) + C,$$

p representing in this equation the atmospheric pressure. Let z denote the value of x at the lowest point of the free surface of the fluid: then

$$p = -\rho g z + C,$$

and therefore $$\omega^2 y^2 = 2g(x - z) \;\ldots\ldots\ldots\ldots\; (2),$$

the equation to the free surface of the fluid.

We proceed now to determine the value of ω and the initial value of z. Let a denote the length of the axis of the vessel, and l the latus rectum of the generating parabola. Then, initially, at the curve of intersection of the vessel and the surface of the fluid, $$\omega^2 l a = 2g(a - z).$$

Also, observing that the volume of a paraboloid of revolution is equal to half that of the circumscribing cylinder, we see that the initial volume of the fluid is equal to

$$\tfrac{1}{2}\pi \, . \, la \, . \, a - \tfrac{1}{2}\pi \, . \, la \, . \, \frac{\omega^2 la}{2g}$$

$$= \tfrac{1}{2}\pi la^2 \left(1 - \frac{\omega^2 l}{2g}\right):$$

but, by the hypothesis, the volume is also equal to

$$\tfrac{1}{2}\pi \, . \, l\,\frac{a}{2} \, . \, \frac{a}{2} = \tfrac{1}{8}\pi la^2:$$

hence

$$\tfrac{1}{2}\pi la^2 \left(1 - \frac{\omega^2 l}{2g}\right) = \tfrac{1}{8}\pi la^2,$$

$$4\left(1 - \frac{\omega^2 l}{2g}\right) = 1, \quad \omega^2 = \frac{3g}{2l}.$$

The equation to the free surface of the fluid will accordingly be, at any time during the motion,

$$y^2 = \tfrac{4}{3} l\,(x - z).$$

Again, at the beginning of the motion, putting a, la, for x, y^2, respectively, we have

$$la = \tfrac{4}{3} l\,(a - z),$$

and therefore $\quad 3a = 4a - 4z, \quad z = \tfrac{1}{4} a.$

At the curve of intersection of the free surface of the fluid and the vessel, at any instant of the motion, there is

$$lx = \tfrac{4}{3} l\,(x - z),$$

$$3x = 4x - 4z, \quad x = 4z, \quad x - z = 3z.$$

Hence, A denoting the volume of the fluid in the vessel at this instant, and x representing $4z$,

$$A = \tfrac{1}{2}\pi \, . \, lx \, . \, x - \tfrac{1}{2}\pi \, . \, lx \, . \, 3z$$

$$= \tfrac{1}{2}\pi l \, . \, 4z \, . (4z - 3z) = 2\pi l z^2.$$

But, v representing the velocity of the issuing fluid, it is plain that

$$kvdt = - dA,$$

and therefore

$$dt = -\frac{4\pi l}{k} \, . \, \frac{zdz}{v} \, . \, . \, . \, . \, . \, . \, . (3).$$

But, from (1), p denoting the atmospheric pressure, there is, at the surface, u^2 being neglected as small,

$$p = \rho\left(\tfrac{1}{2}\omega^2 y^2 - gx\right) + C,$$

or, by (2),

$$p = -\rho g z + C:$$

and, v denoting the velocity of efflux, we have, putting $x = 0$, $y = 0$,

$$p = -\tfrac{1}{2}\rho v^2 + C:$$

from the two last equations we see that

$$v^2 = 2gz.$$

Hence, putting the value of v here given in the equation (3),

$$dt = -\frac{4\pi l}{k} \cdot \frac{z\,dz}{(2gz)^{\frac{1}{2}}}$$

$$= -\frac{4\pi l}{k(2g)^{\frac{1}{2}}} \cdot z^{\frac{1}{2}} dz,$$

and therefore, integrating from $z = \frac{1}{4}a$ to $z = 0$, we obtain for the whole time T of efflux,

$$\frac{2}{3} \cdot \frac{4\pi l}{k(2g)^{\frac{1}{2}}} \cdot z^{\frac{3}{2}} = \frac{8\pi l}{3k(2g)^{\frac{1}{2}}} \cdot \frac{a^{\frac{3}{2}}}{8}$$

$$= \frac{\pi l a^{\frac{3}{2}}}{3k(2g)^{\frac{1}{2}}}.$$

Again, the time T' of efflux, when the fluid is devoid of rotation, is equal to

$$-\frac{1}{k(2g)^{\frac{1}{2}}}\int_{\frac{1}{2}a}^{0} \frac{\pi l z\,dz}{z^{\frac{1}{2}}}$$

$$= \frac{2\pi l}{3k(2g)^{\frac{1}{2}}} \cdot z^{\frac{3}{2}} = \frac{2\pi l}{3k(2g)^{\frac{1}{2}}} \cdot \frac{a^{\frac{3}{2}}}{\sqrt{8}}$$

$$= \frac{\pi l a^{\frac{3}{2}}}{3k\sqrt{2}(2g)^{\frac{1}{2}}}.$$

Hence

$$\frac{T}{T'} = \sqrt{2},$$

which is the required relation.

2. A given quantity of fluid in a vertical cylinder revolves about the axis of the cylinder with a given angular velocity in a form of equilibrium: supposing a small orifice to be made in

the side of the vessel, to determine the interval which will elapse before the lowest point of the revolving fluid descends to the level of the orifice.

Let x denote the altitude of any molecule of the fluid above the orifice, and y the distance of this molecule from the axis of the cylinder. Then, u denoting the component of the velocity of this molecule which is not due to the rotation, and ω the angular velocity of the revolution of the fluid,

$$p = \rho\left(\tfrac{1}{2}\omega^2 y^2 - gx\right) - \tfrac{1}{2}\rho u^2 + C\ldots\ldots\ldots(1).$$

At the surface, u may be regarded as of inconsiderable magnitude, and accordingly, Π denoting the atmospheric pressure,

$$\Pi = \rho\left(\tfrac{1}{2}\omega^2 y^2 - gx\right) + C.$$

If z be the value of x at the lowest point of the surface of the fluid after any time t from the commencement of the efflux,

$$\Pi = -\rho g z + C\ldots\ldots\ldots(2),$$

and therefore we have for the equation to the generating parabola of the surface of the fluid,

$$y^2 = \frac{2g}{\omega^2}(x - z)\ldots\ldots\ldots(3).$$

If v denote the velocity of the issuing fluid, we have, by (1), a denoting the radius of the cylinder,

$$\Pi = \tfrac{1}{2}\rho\omega^2 a^2 - \tfrac{1}{2}\rho v^2 + C,$$

and consequently, by (2),

$$v^2 = \omega^2 a^2 + 2gz\ldots\ldots\ldots(4).$$

If A denote the volume of the fluid in the vessel at the time t, above the horizontal plane through the orifice, then, observing that the volume of a paraboloid of revolution is equal to half that of its circumscribing cylinder,

$$A = \pi a^2 x' - \tfrac{1}{2}\pi a^2 . (x' - z)$$
$$= \pi a^2 z + \tfrac{1}{2}\pi a^2 (x' - z),$$

x' denoting the value of x in the equation (3) when $y = a$; and therefore, by (3),

$$A = \pi a^2 z + \frac{\pi\omega^2 a^4}{4g}\ldots\ldots\ldots(5).$$

But it is evident that

$$kvdt = -dA;$$

hence, by (4) and (5),

$$dt = -\frac{\pi a^2}{k} \cdot \frac{dz}{(\omega^2 a^2 + 2gz)^{\frac{1}{2}}},$$

and accordingly, z' denoting the initial value of z, the required time will be equal to

$$-\frac{\pi a^2}{k}\int_{z'}^{0} \frac{dz}{(\omega^2 a^2 + 2gz)^{\frac{1}{2}}}$$

$$= \frac{\pi a^2}{kg}\left\{(\omega^2 a^2 + 2gz')^{\frac{1}{2}} - \omega a\right\}.$$

Let h denote the length of the portion of the cylinder of which the volume is equal to that of the initial quantity of the fluid. Then, from (5),

$$h = z' + \frac{\omega^2 a^2}{4g};$$

and therefore the expression for the required time is equal to

$$\frac{\pi a^2}{kg}\left\{(2gh + \tfrac{1}{2}\,\omega^2 a^2)^{\frac{1}{2}} - \omega a\right\}.$$

Moseley's *Hydrostatics*, p. 154.

3. To find the time in which a vertical prismatic vessel, full of fluid, will empty itself through a small orifice in its base into a vacuum, its upper surface being exposed to the pressure of the atmosphere.

Let K represent the area of a transverse section of the vessel, h the altitude of the vessel, and h' the altitude of a column of the fluid the weight of which is equal to the atmospheric pressure on an area equal to the base of the column.

Then, x denoting the altitude of any horizontal section of the fluid after any time t above the base of the vessel, and u the velocity of the section, we have

$$p = -\rho\int g dx - \tfrac{1}{2}\rho u^2 + C$$

$$= -g\rho x - \tfrac{1}{2}\rho u^2 + C.$$

Hence, V denoting the velocity at the surface, we have, putting $p = g\rho h'$, and taking x to represent the depth of the fluid,

$$g\rho h' = -g\rho x - \tfrac{1}{2}\rho V^2 + C:$$

at the orifice, v representing the velocity of the issuing fluid,

$$0 = -\tfrac{1}{2}\rho v^2 + C.$$

From the last two equations

$$v^2 - V^2 = 2g(x + h'),$$

or, neglecting V^2 in comparison with v,

$$v = (2g)^{\frac{1}{2}} \cdot (x + h')^{\frac{1}{2}}.$$

But

$$kv = K.V = -K\frac{dx}{dt}:$$

hence

$$\frac{dx}{dt} = -\frac{k}{K}(2g)^{\frac{1}{2}} \cdot (x + h')^{\frac{1}{2}},$$

$$dt = -\frac{K}{k(2g)^{\frac{1}{2}}} \cdot \frac{dx}{(x + h')^{\frac{1}{2}}},$$

and therefore the whole time of efflux will be equal to

$$-\frac{K}{k(2g)^{\frac{1}{2}}}\int_h^0 \frac{dx}{(x + h')^{\frac{1}{2}}}$$

$$= \frac{2K}{k(2g)^{\frac{1}{2}}} \cdot \left\{(h + h')^{\frac{1}{2}} - h'^{\frac{1}{2}}\right\}.$$

Encycl. Metrop. Mix. Sc., vol. I. p. 205.

SECTION III.

Finite Horizontal Orifices. Uniform Motion.

If a vessel, with a finite horizontal aperture, be continually supplied with fluid, so that its surface remains stationary, the velocity of the fluid will be constantly the same at each point of the vessel, and, accordingly, the equation of steady motion will be applicable.

1. Fluid is supplied with a given uniform velocity to a vessel in the form of a truncated paraboloid with its axis vertical: to determine the position of the surface of the fluid that it may remain stationary.

If a denote the distance of the aperture, and x of the surface, from the vertex of the paraboloid, I the volume of the influx of fluid in a unit of time, and l the latus rectum,

$$x = \gamma + (\gamma^2 + 2a\gamma)^{\frac{1}{2}},$$

where

$$\gamma = \frac{I^2}{4\pi^2 l^2 a^2 g}.$$

Moseley's *Hydrostatics*, p. 165.

2. To determine where a semi-ellipsoidal vessel must be truncated by a horizontal plane parallel to the area of its rim, a principal section of the ellipsoid, that, the vessel being kept constantly full, the efflux in a given time may be the greatest possible.

If c denote the vertical semi-axis of the ellipsoid, and z the distance of the required section from the plane of the rim of the vessel;

$$z^4 - 3c^2z^2 - 2c^4 = 0.$$

Moseley's *Hydrostatics*, p. 165.

3. To determine, under the circumstances of the preceding problem, the position of the section that the velocity of efflux may be the greatest possible.

The value of z is given by the equation

$$z^2 = \tfrac{2}{3}c^2.$$

Moseley's *Hydrostatics*, p. 166.

SECTION IV.

Finite Vertical Orifices. Uniform Motion.

Suppose a vessel, in a side of which there is a vertical aperture of finite dimensions, to be supplied with fluid at such a rate that its surface may remain stationary. Let λ represent the area of an element of the aperture, v the velocity of the fluid issuing through this element, K the area of the surface,

and V the velocity of the molecules of fluid at the surface. Then, the influx being, by the condition of the problem, equal to the efflux, we shall have

$$KV = \Sigma(v\lambda).$$

Also, x denoting the depth of the elemental area λ below the surface,

$$v^2 = V^2 + 2gx.$$

The expression KV represents the volume of the influx in a unit of time. From the two equations here laid down the rate of influx may be ascertained when the position of the surface is assigned, or the converse. The truth of the latter of the two equations depends upon the supposition that the motion is steady, a condition which is secured by supposing the influx to be sufficient to keep the surface stationary.

1. To determine the quantity of fluid which will escape in a given time through a rectangular vertical orifice $LMNO$, in the side of a vessel $ABCD$ kept full of the fluid, the magnitude of the aperture, although finite, being inconsiderable in comparison with a horizontal section of the vessel.

Let h, h', (fig. 34) represent the depths of LO, MN, respectively, below the surface AD of the fluid, and a the breadth LO of the aperture. Also let x denote the depth of a horizontal line mn in the aperture, below AD. Then, V being, by the nature of the problem, inconsiderable in comparison with v, the whole efflux in a time t will be equal to

$$\begin{aligned}
&\int_h^{h'}\int_0^t v\,dt\,.\,a\,dx \\
&= a(2g)^{\frac{1}{2}}\int_h^{h'}\int_0^t x^{\frac{1}{2}}\,dt\,dx \\
&= \tfrac{2}{3}a(2g)^{\frac{1}{2}}\int_0^t \left(h'^{\frac{3}{2}} - h^{\frac{3}{2}}\right) dt \\
&= \tfrac{2}{3}at(2g)^{\frac{1}{2}}\left(h'^{\frac{3}{2}} - h^{\frac{3}{2}}\right).
\end{aligned}$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 267.
Encycl. Metrop., Mix. Sc., vol. I. p. 206.

2. To determine the quantity of fluid which will escape in a given time through a circular vertical aperture in a vessel kept constantly full, the size of the aperture being small when compared with the surface of the fluid.

Let r denote the radius of the aperture, nr the depth of the centre of the aperture below the surface of the fluid, θ the inclination of any radius of the aperture to the radius which extends vertically upwards. Then, the area of a strip of the circular aperture included between two consecutive double horizontal ordinates, the length and depth of the higher being $2y$ and x respectively, will be equal to

$$\begin{aligned} 2y\,dx &= -\,2r\sin\theta\, d(r\cos\theta) \\ &= 2r^2\sin^2\theta\, d\theta: \end{aligned}$$

also the velocity of efflux through the strip will be equal to

$$(2gx)^{\frac{1}{2}} = (2g)^{\frac{1}{2}}\ (nr - r\cos\theta)^{\frac{1}{2}}.$$

Hence the quantity of fluid which escapes in a time t will be equal to

$$2\,(2ng)^{\frac{1}{2}}\, r^{\frac{5}{2}}\, t\int_0^{\pi} \sin^2\theta\left(1 - \frac{\cos\theta}{n}\right)^{\frac{1}{2}} d\theta$$

$$= 2\,(2ng)^{\frac{1}{2}}\, r^{\frac{5}{2}}\, t\int_0^{\pi} \sin^2\theta\left(1 - \frac{1}{2}\ \frac{\cos\theta}{n} - \frac{1.1}{1.2}\cdot\frac{\cos^2\theta}{2^2n^2} - \frac{1.1.3}{1.2.3}\cdot\frac{\cos^3\theta}{2^3n^3} \right.$$
$$\left. - \frac{1.1.3.5}{1.2.3.4}\cdot\frac{\cos^4\theta}{2^4n^4} - \dots\right).$$

Now

$$\int_0^{\pi} \sin^2\theta\cos^n\theta\, d\theta = \Big|_0^{\pi}\left(-\frac{1}{n+1}\sin\theta\cos^{n+1}\theta\right) + \frac{1}{n+1}\int_0^{\pi}\cos^{n+2}\theta\, d\theta$$

$$= \frac{1}{n+1}\int_0^{\pi}\cos^n\theta\,(1 - \sin^2\theta)\, d\theta$$

$$= \frac{1}{n+2}\int_0^{\pi}\cos^n\theta\, d\theta$$

$$= \frac{1.3.5\dots(n-1)}{2.4.6\dots\ n}\ \frac{\pi}{n+2},\quad \text{if } n \text{ be even}:$$

$$= 0,\quad \text{if } n \text{ be odd.}$$

Hence the quantity of efflux is equal to

$$2\,(2ng)^{\frac{1}{2}}r^{\frac{5}{2}}t\left\{\frac{\pi}{2}-\frac{1.1}{1.2}\cdot\frac{1}{2^2n^2}\cdot\frac{1}{2}\cdot\frac{\pi}{4}-\frac{1.1.3.5}{1.2.3.4}\cdot\frac{1}{2^4n^4}\cdot\frac{1.3}{2.4}\cdot\frac{\pi}{6}-\dots\right\}$$

$$=\pi\,(2ng)^{\frac{1}{2}}r^{\frac{5}{2}}t\,.\left\{1-\frac{1}{32n^2}-\frac{5}{1024n^4}-\dots\right\}.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 274.

3. A reservoir, kept constantly full, has two triangles, of which the bases are horizontal, cut in its side, the one having its base and the other its vertex upwards: the bases and altitudes of the triangles are equal, and the depth of the base of the former is equal to that of the vertex of the latter: to compare the quantity of fluid discharged by the apertures in the same time, supposing their areas to be small in comparison with the surface of the fluid.

If h, h', be the depths of the highest and lowest part of either triangle, and c the length of the base of either, the volume discharged in the time t by the former triangle will be equal to

$$\frac{2ct\,(2g)^{\frac{1}{2}}}{15\,(h'-h)}\,.\,(2h^{\frac{5}{2}}+3h'^{\frac{5}{2}}-5hh'^{\frac{3}{2}}),$$

and by the latter to

$$\frac{2ct\,(2g)^{\frac{1}{2}}}{15\,(h'-h)}\,.\,(2h'^{\frac{5}{2}}+3h^{\frac{5}{2}}-5h'h^{\frac{3}{2}}),$$

the ratio between the two volumes being therefore equal to

$$\frac{2h^{\frac{5}{2}}+3h'^{\frac{5}{2}}-5hh'^{\frac{3}{2}}}{2h'^{\frac{5}{2}}+3h^{\frac{5}{2}}-5h'h^{\frac{3}{2}}}.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 271.
Encycl. Metrop., Mix. Sc., vol. I. p. 206.

4. There is a vertical rectangular aperture, with two sides vertical, in the side of a vessel containing fluid, of which the surface is kept stationary: to determine the relation between the velocity of influx and the elements of position and magnitude of the aperture, the aperture not being small compared with the surface of the fluid.

If a denote the length of a horizontal and b of a vertical side of the rectangle, and c the depth of its highest side, then

$$\frac{a}{3g}\left[\{V^2 + 2g(c+b)\}^{\frac{3}{2}} - (V^2 + 2gc)^{\frac{3}{2}}\right] = KV.$$

Moseley's *Hydrostatics*, p. 167.

5. A circular vertical aperture is made in the side of a vessel containing fluid, the surface of which is kept constantly at the altitude of the highest point of the aperture: to determine the volume of efflux in a given time, the area of the aperture, although finite, being small compared with the surface of the fluid.

If r denote the radius of the aperture, the volume of the fluid which escapes in a time t is equal to

$$\tfrac{64}{15} g^{\frac{1}{2}} r^{\frac{5}{2}} t.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 276.

SECTION V.

Finite Horizontal Apertures of any magnitude. Variable Motion.

Let x denote the depth of the surface of the fluid at the end of any time t from the commencement of the motion below a fixed horizontal plane, z the depth of any section of the fluid, and c the depth of the orifice. Let K denote the area of the surface at the time t, Z the area of the horizontal section at the depth z, and k the area of the orifice.

Then, by the general equation of fluid motion, ds denoting the element of the path described by any molecule of fluid at the depth z below the fixed plane, $g\dfrac{dz}{ds}$ the component of gravity along ds, p representing the pressure at the molecule and u its velocity,

$$\frac{du}{dt} + u\frac{du}{ds} = g\frac{dz}{ds} - \frac{1}{\rho}\frac{dp}{ds};$$

or, adopting the hypothesis of *parallel sections*, in which it is supposed, as approximately true, that the molecules of the fluid all descend vertically, and that each horizontal slice of the fluid subsides into the position before occupied by the slice next lower, we have, putting dz instead of ds, and taking u to represent the velocity of any point whatever in the horizontal section Z,

$$\frac{du}{dt} + u\frac{du}{dz} = g - \frac{1}{\rho}\frac{dp}{dz} \quad \ldots\ldots\ldots\ldots (1).$$

Now from the hypothesis of parallel sections it follows that, v being the velocity of the issuing fluid,

$$kv = Zu \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and therefore, $\frac{du}{dt}$ denoting the partial differential coefficient of u on the hypothesis that t varies while z remains constant, we have

$$k\frac{dv}{dt} = Z\frac{du}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we obtain

$$k\frac{dv}{dt}\frac{1}{Z} + u\frac{du}{dz} = g - \frac{1}{\rho}\frac{dp}{dz},$$

and therefore, by integration, observing that $\frac{dv}{dt}$ is independent of z, which is merely an arbitrary quantity in regard to the efflux, we have

$$k\frac{dv}{dt}\int\frac{dz}{Z} + \tfrac{1}{2}u^2 = gz - \frac{1}{\rho}p.$$

Let p' denote the unit of pressure at the surface: then, V denoting the velocity of the particles of the surface,

$$k\frac{dv}{dt}\int_x^z\frac{dz}{Z} + \tfrac{1}{2}(u^2 - V^2) = g(z - x) - \frac{1}{\rho}(p - p') \ldots\ldots (4);$$

and therefore, since Zu and KV are each of them equal to kv,

$$\frac{1}{\rho}(p - p') = g(z - x) - k\frac{dv}{dt}\int_x^z\frac{dz}{Z} + \tfrac{1}{2}k^2v^2\left(\frac{1}{K^2} - \frac{1}{Z^2}\right) \ldots (5).$$

Again, since v and x are both functions of a single variable t,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = V\frac{dv}{dx} = \frac{k}{K}v\frac{dv}{dx}:$$

hence our equation may be transformed into the following,

$$\frac{1}{\rho}(p-p') = g(z-x) - \frac{k^2}{K}v\frac{dv}{dx}\int_x^z \frac{dz}{Z} + \tfrac{1}{2}k^2v^2\left(\frac{1}{K^2} - \frac{1}{Z^2}\right)\dots(6).$$

Let p'' denote the value of p at the orifice: then, from (6),

$$\frac{1}{\rho}(p''-p') = g(c-x) - \frac{k^2}{K}v\frac{dv}{dx}\int_x^c \frac{dz}{Z} - \tfrac{1}{2}v^2\left(1 - \frac{k^2}{K^2}\right)\dots(7).$$

From the equation (7), which is linear in regard to v^2, v may be obtained in terms of x, and then, from the equation

$$KV = kv,$$

or

$$K\frac{dx}{dt} = kv,$$

x and $\frac{dx}{dt}$ or V may be obtained in terms of t. The equation (6), v and x having been accordingly expressed in terms of t, will give p in terms of z and t. The equation (2), v being known in terms of t, will give u in terms of z and t.

1. To investigate the rate of efflux from a vessel through a horizontal aperture when the surface of the contained fluid is maintained at a constant altitude by the perpetual superposition of fresh fluid, the aperture being not greater than the surface of the fluid.

Putting p'' for p, and c for z, in the equation (5), we have

$$\frac{1}{\rho}(p''-p') = g(c-x) - k\frac{dv}{dt}\int_x^c \frac{dz}{Z} - \tfrac{1}{2}v^2\left(1 - \frac{k^2}{K^2}\right).$$

Hence, λ and μ being constant quantities,

$$dv + \lambda^2v^2\,dt = \mu^2\,dt,$$

$$dt = \frac{1}{\lambda^2}\,\frac{dv}{\dfrac{\mu^2}{\lambda^2} - v^2}$$

$$= \frac{1}{2\lambda\mu}\left\{\frac{1}{\dfrac{\mu}{\lambda} - v} + \frac{1}{\dfrac{\mu}{\lambda} + v}\right\},$$

$$t = \frac{1}{2\lambda\mu} \log \left\{ \frac{\frac{\mu}{\lambda} + v}{\frac{\mu}{\lambda} - v} \right\},$$

$$\frac{\frac{\mu}{\lambda} + v}{\frac{\mu}{\lambda} - v} = \epsilon^{2\lambda\mu t},$$

$$v = \frac{\mu}{\lambda} \cdot \frac{\epsilon^{2\lambda\mu t} - 1}{\epsilon^{2\lambda\mu t} + 1}.$$

As t increases without limit, the expression

$$\frac{\epsilon^{2\lambda\mu t} - 1}{\epsilon^{2\lambda\mu t} + 1}$$

ultimately becomes equal to unity, and therefore ultimately

$$v = \frac{\mu}{\lambda}.$$

But
$$\lambda^2 = \frac{\frac{1}{2}\left(1 - \frac{k^2}{K^2}\right)}{k\int_x^c \frac{dz}{Z}},$$

and
$$\mu^2 = \frac{g(c-x) + \frac{1}{\rho}(p' - p'')}{k\int_x^c \frac{dz}{Z}};$$

hence
$$v^2 = \frac{\mu^2}{\lambda^2} = \frac{2g(c-x) + \frac{2}{\rho}(p' - p'')}{1 - \frac{k^2}{K^2}},$$

which gives the value of v when the motion becomes steady.

Bossut: *Traité d'Hydrodynamique*, tom. I., p. 286.

2. Under the circumstances of the preceding problem, to determine the pressure at any point in the fluid at any instant of the motion.

In order to obtain the expression for p, we must substitute the value of v, viz.

$$\frac{\mu}{\lambda} \frac{\epsilon^{2\lambda\mu t}-1}{\epsilon^{2\lambda\mu t}+1},$$

in the formula

$$\frac{1}{\rho}(p-p') = g(z-x) - k\frac{dv}{dt}\int_a^z \frac{dz}{Z} + \tfrac{1}{2}k^2v^2\left(\frac{1}{K^2}-\frac{1}{Z^2}\right).$$

Now, performing the operation of differentiation, we shall get

$$\frac{dv}{dt} = \frac{4\mu^2\epsilon^{2\lambda\mu t}}{(\epsilon^{2\lambda\mu t}+1)^2}.$$

Hence $\frac{1}{\rho}(p-p')$

$$= g(z-x) + \frac{\mu^2 k}{(\epsilon^{2\lambda\mu t}+1)^2}\left\{\frac{k}{2\lambda^2}(\epsilon^{2\lambda\mu t}-1)^2 \cdot \left(\frac{1}{K^2}-\frac{1}{Z^2}\right) - 4\epsilon^{2\lambda\mu t}\int_a^z \frac{dz}{Z}\right\},$$

which determines the value of p.

When $t=\infty$, that is, when the motion becomes steady,

$$\frac{\epsilon^{2\lambda\mu t}}{(\epsilon^{2\lambda\mu t}+1)^2} = 0, \quad \frac{\epsilon^{2\lambda\mu t}-1}{\epsilon^{2\lambda\mu t}+1} = 1,$$

and therefore

$$\frac{1}{\rho}(p-p') = g(z-x) + \frac{\mu^2k^2}{2\lambda^2}\left(\frac{1}{K^2}-\frac{1}{Z^2}\right).$$

3. To determine the velocity of efflux from a horizontal aperture of any magnitude in the base of a vertical cylinder containing fluid, the surface of the fluid and the aperture being both exposed to the atmosphere.

Since $p' = p''$, we have, from the equation (7),

$$\frac{k^2}{K}v\frac{dv}{dx}\int_a^c \frac{dz}{Z} + \tfrac{1}{2}v^2\left(1-\frac{k^2}{K^2}\right) = g(c-x).$$

In this problem Z is equal to K, and therefore

$$\frac{k^2}{K^2}(c-x)\,v\frac{dv}{dx} + \tfrac{1}{2}v^2\left(1-\frac{k^2}{K^2}\right) = g(c-x),$$

or, putting $K^2 = m\,k^2$,

$$2v\,dv + (m-1)\frac{v^2dx}{c-x} = 2mg\,dx,$$

$$d\,(Pv^2) = 2mgP\,dx,$$

where $$P = \epsilon^{\int \frac{m-1}{c-x}dx} = \epsilon^{\log (c-x)^{1-m}} = (c-x)^{1-m}:$$

hence $$(c-x)^{1-m}\,v^2 = 2mg\int (c-x)^{1-m}\,dx$$

$$= \frac{2mg}{m-2}(c-x)^{2-m} + C,$$

C being an arbitrary constant,

or $$v^2 = \frac{2mg}{m-2}(c-x) + C\,(c-x)^{m-1}.$$

Let the origin of x be so chosen that x and v may be zero simultaneously: then

$$0 = \frac{2mg}{m-2}c + Cc^{m-1},$$

$$C = -\frac{2mg}{m-2}c^{2-m},$$

and therefore for the velocity of efflux there is

$$v^2 = \frac{2mg}{m-2}\{c - x - c^{2-m}(c-x)^{m-1}\}.$$

If $K^2 = 2k^2$, or $m = 2$, v^2 assumes the indeterminate form $\frac{0}{0}$. In this case, adopting the ordinary method for the evaluation of vanishing fractions, we have

$$\frac{c-x-c^{2-m}(c-x)^{m-1}}{m-2} = \frac{\log c\,.\,c^{2-m}.\,(c-x)^{m-1} - \log(c-x)\,.\,c^{2-m}.\,(c-x)^{m-1}}{1}$$

$$= \log c\,.\,(c-x) - \log(c-x)\,.\,(c-x)$$

$$= (c-x)\log\frac{c}{c-x},$$

and therefore $$v^2 = 4g\,(c-x)\log\frac{c}{c-x}.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 291—293.

4. Under the circumstances of the preceding problem, to determine the pressure at any point of the fluid corresponding to any position of the surface of the fluid.

The general formula for the pressure is

$$\frac{1}{\rho}(p-p')=g(z-x)-\frac{k^2}{K}v\frac{dv}{dx}\int_x^z\frac{dz}{Z}+\tfrac{1}{2}k^2v^2\left(\frac{1}{K^2}-\frac{1}{Z^2}\right),$$

or, putting $Z=K$,

$$\frac{1}{\rho}(p-p')=g(z-x)-\frac{k^2}{K^2}v\frac{dv}{dx}(z-x).$$

But, unless $K^2=2k^2$, we have

$$v^2=\frac{2mg}{m-2}\{c-x-c^{2-m}(c-x)^{m-1}\},$$

$$v\frac{dv}{dx}=\frac{mg}{m-2}\{(m-1)c^{2-m}(c-x)^{m-2}-1\}.$$

Hence, remembering that $K^2=mk^2$,

$$\frac{1}{\rho}(p-p')=g(z-x)-\frac{g}{m-2}\{(m-1)c^{2-m}(c-x)^{m-2}-1\}(z-x)$$

which determines the pressure at any depth within the fluid when the surface is in any assignable position.

If $K^2=2k^2$, or $m=2$, then

$$v^2=4g(c-x)\log\frac{c}{c-x},$$

and $$v\frac{dv}{dx}=2g\left(1+\log\frac{c-x}{c}\right):$$

hence $$\frac{1}{\rho}(p-p')=g(z-x)-g\left(1+\log\frac{c-x}{c}\right)(z-x).$$

SECTION VI.

*Small Finite Vertical Apertures. Variable Motion.**

If fluid be not supplied to the vessel, the velocity of the efflux will vary with the variation of the altitude of the fluid. The method of determining the rate of efflux in problems of this class depends upon the supposition that the surface may be regarded as stationary, and accordingly the motion as steady for each small element of time, the aperture being small in comparison with the surface of the fluid, although so large that all its points cannot be considered to be at the same depth.

Ex. 1. To determine in what time the surface of the fluid in a vertical prismatic vessel will be depressed through a given space by the efflux of the fluid through a vertical rectangular aperture having two of its sides horizontal, the magnitude of the aperture being small in comparison with the surface.

Let K denote the area of a horizontal section of the prism, z the depth of the surface of the fluid below its initial position after a time t, h the depth of the upper, h' of the lower side of the aperture, and x of any point in the aperture, below the initial position of the surface; b the length of a horizontal side of the aperture.

Then
$$K\frac{dz}{dt} = \int_{h-z}^{h'-z} (2gx)^{\frac{1}{2}}\, b\, dx$$
$$= \tfrac{2}{3}(2g)^{\frac{1}{2}}\, b\, \{(h'-z)^{\frac{3}{2}} - (h-z)^{\frac{3}{2}}\},$$

and therefore, c denoting the given depth of the depression of the surface of the fluid,

$$t = \frac{3K}{2(2g)^{\frac{1}{2}}b}\int_0^c \frac{dz}{(h'-z)^{\frac{3}{2}} - (h-z)^{\frac{3}{2}}}$$
$$= \frac{3K}{2(2g)^{\frac{1}{2}}b}\int_0^c dz\,\frac{(h'-z)^{\frac{3}{2}} + (h-z)^{\frac{3}{2}}}{(h'-z)^3 - (h-z)^3}$$
$$= \frac{3K}{2(2g)^{\frac{1}{2}}b(h'-h)}\int_0^c dz\,\frac{(h-z)^{\frac{3}{2}} + (h'-z)^{\frac{3}{2}}}{h^2 + hh' + h'^2 - 3(h+h')z + 3z^2}.$$

* D'Alembert: *Traité des fluides*, p. 113. *Opuscules Mathématiques*, tom. I. pp. 159, 160.

The evaluation of the two integrals

$$\int_0^c \frac{(h - z)^{\frac{3}{2}}\, dz}{h^2 + hh' + h'^2 - 3(h + h')z + 3z^2},$$

$$\int_0^c \frac{(h' - z)^{\frac{3}{2}}\, dz}{h^2 + hh' + h'^2 - 3(h + h')z + 3z^2},$$

by substituting y^2 for $h - z$ in the former, and y'^2 for $h' - z$ in the latter, may be reduced to the integration of rational fractions, and may be effected without difficulty by the common rules.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 279.
Encycl. Metrop. Mix. Sc., vol. I. p. 207.

SECTION VII.

Motion of Fluids through a system of communicating vessels.

1. Three open vessels, $ABCD$, $FCEG$, $HELK$ (fig. 35), communicate with each other by small apertures C, E, near the horizontal plane of their bases, and the fluid escapes into the air through the small aperture L in the last of them: to determine the velocities of the fluid at the three apertures, when the motion has become steady, the surfaces $A'D'$, $F'G'$, $H'K'$, being constantly sustained at the same altitudes by the influx of fluid into the vessel $ABCD$.

Let $A'B = h$, $D'F' = x$, $G'H' = x'$, $K'L = x''$: let k, k', k'', represent the orifices C, E, L, and Q the volume of the fluid which escapes through L in a time t. Also let v, v', v'', denote the velocities at the three orifices.

Now the velocity of the fluid at C is due to the difference of the altitudes of the fluids in the two vessels $ABCD$, $FCEG$, since the fluid $CF'G'E$ may be regarded as producing equilibrium with the portion of the fluid $A'BCD'$, which is of the same altitude above the base BC.

Hence $\qquad v = (2gx)^{\frac{1}{2}}$,

and similarly $\qquad v' = (2gx')^{\frac{1}{2}}, \qquad v'' = (2gx'')^{\frac{1}{2}}$.

The fluid which passes through the three orifices in a time t will be equal to

$$(2gx)^{\frac{1}{2}}.\,kt, \quad (2gx')^{\frac{1}{2}}.\,k't, \quad (2gx'')^{\frac{1}{2}}.\,k''t.$$

Since these must be all equal, we have

$$x^{\frac{1}{2}}k = x'^{\frac{1}{2}}k' = x''^{\frac{1}{2}}k''.$$

But $$x + x' + x'' = h:$$

hence $$k^2x = k'^2x' = k''^2x'' = \frac{h}{\dfrac{1}{k^2} + \dfrac{1}{k'^2} + \dfrac{1}{k''^2}},$$

and, accordingly,

$$kv = k'v' = k''v'' = \left\{\frac{2gh}{\dfrac{1}{k^2} + \dfrac{1}{k'^2} + \dfrac{1}{k''^2}}\right\}^{\frac{1}{2}},$$

which equations determine the velocities.

Also, for the volume of efflux in any time, we have

$$Q = vkt = \left\{\frac{2ght^2}{\dfrac{1}{k^2} + \dfrac{1}{k'^2} + \dfrac{1}{k''^2}}\right\}^{\frac{1}{2}}.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 328.

2. The two vertical prismatic vessels $ABCD$, $FCEG$, (fig. 36) communicate together by the small orifice C, and the latter discharges fluid through a small orifice at E, while the former is kept constantly full: to determine the position of the surface of the fluid in $FCEG$ at the end of any proposed time.

Let NO be the position of the surface in the vessel $FCEG$ after a time t: let $CD = h$, $DN = x$, $C =$ the area of the orifice at C, $E =$ that of the orifice at E, A = the area of a horizontal section of $FCEG$.

Then, $-dx$ denoting the elevation of NO in a time dt,

$$-Adx = C(2gx)^{\frac{1}{2}}dt - E\{2g(h-x)\}^{\frac{1}{2}}.\,dt,$$

$$dt = \frac{A}{(2g)^{\frac{1}{2}}} \cdot \frac{dx}{E(h-x)^{\frac{1}{2}} - Cx^{\frac{1}{2}}}.$$

Putting $$x = \frac{y^2}{h},$$

and $$E(h^2 - y^2)^{\frac{1}{2}} - Cy = Ez,$$

this equation will be transformed into the following one,

$$dt = Mdz + \frac{Nz\,dz}{(P^2 - z^2)^{\frac{1}{2}}} + \frac{Qdz}{z\,(P^2 - z^2)^{\frac{1}{2}}},$$

$M, N, P, Q,$ representing constant quantities; hence

$$t = Mz - N(P^2 - z^2)^{\frac{1}{2}} - \frac{Q}{P}\log\left\{\frac{1}{z} + \left(\frac{1}{z^2} - \frac{1}{P^2}\right)^{\frac{1}{2}}\right\} + \text{const.}$$

This equation establishes the relation between the position of the surface and the time.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 330.

3. A vertical prismatic vessel $FCEG$, (fig. 37) supposed to be uniformly supplied with fluid, discharges a portion of the supply through a small aperture at E: to determine the time in which the surface of the fluid will rise to any proposed altitude.

Let C denote the volume of the fluid supplied to the vessel in a unit of time; and k the area of the effective orifice at E. Then, x denoting the altitude NC of the fluid, and K the area of a horizontal section of the vessel,

$$Kdx = Cdt - k(2gx)^{\frac{1}{2}}\,dt,$$

$$dt = K\,\frac{dx}{C - k(2g)^{\frac{1}{2}} . x^{\frac{1}{2}}},$$

or, putting $x = y^2$,

$$dt = 2K\,\frac{y\,dy}{C - (2g)^{\frac{1}{2}}\,ky}$$

$$= -\frac{2K}{(2g)^{\frac{1}{2}}k} \cdot \frac{C - (2g)^{\frac{1}{2}}ky}{C - (2g)^{\frac{1}{2}}ky}\,dy$$

$$+ \frac{2CK}{(2g)^{\frac{1}{2}}k} \cdot \frac{dy}{C - (2g)^{\frac{1}{2}}ky},$$

$$t = -\frac{2Ky}{(2g)^{\frac{1}{2}}k} - \frac{CK}{gk^2}\log\{C - (2g)^{\frac{1}{2}}ky\} + \text{const.}$$

$$= -\frac{2Kx^{\frac{1}{2}}}{(2g)^{\frac{1}{2}}k} - \frac{CK}{gk^2}\log\{C - k(2gx)^{\frac{1}{2}}\} + \text{const.}$$

Let $x = a$, when $t = 0$: then

$$t = \frac{K}{k}\left\{\left(\frac{2a}{g}\right)^{\frac{1}{2}} - \left(\frac{2x}{g}\right)^{\frac{1}{2}}\right\} - \frac{CK}{gk^2}\log\left\{\frac{C - k(2gx)^{\frac{1}{2}}}{C - k(2ga)^{\frac{1}{2}}}\right\}.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 333.

4. The vessel $ABCD$ (fig. 38), supposed to be prismatic and vertical, has a small orifice G in its base, and is divided into two compartments by a horizontal diaphragm EF in which there is a small orifice H: supposing the vessel to be kept constantly full of fluid at the level AD, while efflux takes place through the two orifices; to determine the circumstances of the rate of discharge at H and G.

Let k_1, k_2, represent the orifices at H, G: let $AB = h$, $AE = x_1$; $EB = x_2$; v_1 = the velocity at H, v_2 = the velocity at G.

If the efflux through G exceed that through H, the fluids in the two compartments of the vessel will separate from each other, a vacuum being thus created between the diaphragm and the surface of the fluid in the lower vessel. Under these circumstances, both compartments being filled, we shall have initially, Π denoting the atmospheric pressure, and ρ the density of the fluid,

$$v_1^2 = 2gx_1 + \frac{2\Pi}{\rho}, \quad v_2^2 = 2gx_2 - \frac{2\Pi}{\rho},$$

$$k_1^2v_1^2 = k_1^2\left(2gx_1 + \frac{2\Pi}{\rho}\right), \quad k_2^2v_2^2 = k_2^2\left(2gx_2 - \frac{2\Pi}{\rho}\right),$$

and therefore, since k_1v_1 is supposed to be less than k_2v_2,

$$k_1^2\left(gx_1 + \frac{\Pi}{\rho}\right) < k_2^2\left(gx_2 - \frac{\Pi}{\rho}\right):$$

this inequality expresses the condition that the fluid in the two compartments may separate.

We will however suppose that the condition for the separation of the fluid into two parts does not hold. In this case, p denoting the pressure at H, and y the depth of any point in the upper compartment below AD,

$$p - \Pi = g\rho\int_0^{x_1} dy - \tfrac{1}{2}\rho v_1^2$$
$$= g\rho x_1 - \tfrac{1}{2}\rho v_1^2:$$

and, z denoting the depth of any point in the lower compartment below EF,

$$\Pi - p = g\rho \int_0^{x_2} dz - \tfrac{1}{2}\rho v_2^2$$

$$= g\rho x_2 - \tfrac{1}{2}\rho v_2^2.$$

Adding together these two equations, we obtain

$$0 = g(x_1 + x_2) - \tfrac{1}{2}(v_1^2 + v_2^2),$$

$$v_1^2 + v_2^2 = 2g(x_1 + x_2) = 2gh.$$

Since the efflux through the two orifices must be the same, we have also

$$k_1 v_1 = k_2 v_2;$$

hence

$$v_1^2 = \frac{2gh}{k_1^2} \cdot \frac{1}{\dfrac{1}{k_1^2} + \dfrac{1}{k_2^2}},$$

$$v_2^2 = \frac{2gh}{k_2^2} \cdot \frac{1}{\dfrac{1}{k_1^2} + \dfrac{1}{k_2^2}}.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 341.

5. A vertical prismatic vessel $ACKB$, (fig. 39) which communicates with a tube KL, closed on all sides, except at the point D, where there is a small orifice, is traversed by several horizontal diaphragms EF, OP, VH, in which small orifices G, M, N, are pierced: to find the time in which the surface of the fluid will descend from AB to the level TL, the vessel discharging fluid through the orifice D without receiving any fresh supply. The fluids in the several compartments are supposed not to detach themselves from each other.

Let $BF = x_1$, $FP = x_2$, $PH = x_3$, $HT = x_4$; let k_1, k_2, k_3, k_4, denote the areas of the orifices at G, M, N, D, and v_1, v_2, v_3, v_4, the velocities of the fluid through these orifices at a time t, ab being the position of the surface of the fluid at this time. Let $bF = y_1$; let Π denote the atmospheric pressure, and p_1, p_2, p_3, the pressures at G, M, N. Then

$$p_1 - \Pi = g\rho y_1 - \tfrac{1}{2}\rho v_1^2,$$

$$p_2 - p_1 = g\rho x_2 - \tfrac{1}{2}\rho v_2^2,$$

$$p_3 - p_2 = g\rho x_3 - \tfrac{1}{2}\rho v_3^2,$$

$$\Pi - p_3 = g\rho x_4 - \tfrac{1}{2}\rho v_4^2;$$

and consequently

$$0 = g(y_1 + x_2 + x_3 + x_4) - \tfrac{1}{2}(v_1^2 + v_2^2 + v_3^2 + v_4^2),$$

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 = 2g(y_1 + x_2 + x_3 + x_4) \dots\dots (1).$$

Also, the efflux through all the orifices being equal in equal times, $$k_1v_1 = k_2v_2 = k_3v_3 = k_4v_4 = \lambda \dots\dots\dots\dots\dots (2),$$

λ denoting the efflux in a unit of time due to these velocities. From (1) and (2)

$$\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} + \frac{1}{k_4^2} = \frac{2g}{\lambda^2}(y_1 + x_2 + x_3 + x_4),$$

and therefore, from (2),

$$k_4^2v_4^2 = (y_1 + x_2 + x_3 + x_4) \cdot \frac{2g}{\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} + \frac{1}{k_4^2}}$$

$$= n_1^2 \cdot (y_1 + x_2 + x_3 + x_4),$$

where $$n_1^2 = \frac{2g}{\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} + \frac{1}{k_4^2}}.$$

Hence, K denoting the area of a horizontal section of the vessel,

$$K\frac{dy_1}{dt} = -k_4v_4 = -n_1(y_1 + x_2 + x_3 + x_4)^{\frac{1}{2}},$$

$$dt = -\frac{K}{n_1}\frac{dy_1}{(y_1 + x_2 + x_3 + x_4)^{\frac{1}{2}}},$$

and therefore the time in descending from AB to EF is equal to

$$T_1 = \frac{2K}{n_1}(f_1^{\frac{1}{2}} - f_2^{\frac{1}{2}}),$$

f_r denoting $x_r + x_{r+1} + \dots\dots + x_4$.

Similarly, T_2 representing the time through FP, T_3 that through PH, and T_4 that through HT, and n_r denoting

$$\frac{1}{k_r^2} + \frac{1}{k_{r+1}^2} + \dots\dots + \frac{1}{k_4^2},$$

we have $$T_2 = \frac{2K}{n_2}(f_2^{\frac{1}{2}} - f_3^{\frac{1}{2}}),$$

$$T_3 = \frac{2K}{n_3}(f_3^{\frac{1}{2}} - f_4^{\frac{1}{2}}),$$

$$T_4 = \frac{2K}{n_4}(f_4^{\frac{1}{2}}).$$

Hence the whole time required is equal to

$$2K\left\{\frac{f_1^{\frac{1}{2}}-f_2^{\frac{1}{2}}}{n_1}+\frac{f_2^{\frac{1}{2}}-f_3^{\frac{1}{2}}}{n_2}+\frac{f_3^{\frac{1}{2}}-f_4^{\frac{1}{2}}}{n_3}+\frac{f_4^{\frac{1}{2}}}{n_4}\right\}.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 345.

6. The circumstances remaining the same as in the preceding problem, excepting that the densities of the fluids are different in the several compartments, to determine the velocities at the several orifices at any moment of time. The interval which has elapsed from the commencement of the motion is supposed not to be so great as to allow the densities of the fluids to be sensibly deranged by commixture.

Instead of the equations connecting the pressures and velocities given in the preceding problem we shall have, $\rho_1, \rho_2, \rho_3, \ldots$ being the densities of the fluids in order, beginning from the highest,

$$p_1 - \Pi = g\rho_1 y_1 - \tfrac{1}{2}\rho_1 v_1^2,$$
$$p_2 - p_1 = g\rho_2 x_2 - \tfrac{1}{2}\rho_2 v_2^2,$$
$$p_3 - p_2 = g\rho_3 x_3 - \tfrac{1}{2}\rho_3 v_3^2,$$
$$\Pi - p_3 = g\rho_4 x_4 - \tfrac{1}{2}\rho_4 v_4^2;$$

and therefore

$$\rho_1 v_1^2 + \rho_2 v_2^2 + \rho_3 v_3^2 + \rho_4 v_4^2 = 2g\,(\rho_1 y_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4).$$

But λ, denoting the volume of the efflux in a unit of time due to these velocities,

$$k_1 v_1 = k_2 v_2 = k_3 v_3 = k_4 v_4 = \lambda,$$

and therefore $$\lambda^2 = 2g\,\frac{\rho_1 y_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4}{\dfrac{\rho_1}{k_1^2} + \dfrac{\rho_2}{k_2^2} + \dfrac{\rho_3}{k_3^2} + \dfrac{\rho_4}{k_4^2}}.$$

Hence v_1, v_2, v_3, v_4, are known for every position of the descending surface in the highest compartment.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 349.

7. The fluid in a vessel $ABCD$, (fig. 40), kept constantly full at the altitude AB, passes through a small aperture M in the side of a lateral vessel $CEGF$ closed on all sides, except at N and P, where there are two small apertures through which the fluid may escape: to find the velocities of the fluid at M, N, P.

Let v_1, v_2, v_3, be the velocities at M, N, P, the areas of these orifices being k_1, k_2, k_3. Let $AB = a$, and let h denote the depth of N below AD. Let x denote the altitude due to a velocity v_1: then

$$v_1^2 = 2gx,$$

$$v_2^2 = 2g(h - x),$$

$$v_3^2 = 2g(a - x).$$

But, the efflux through M being equal to the efflux through N and P together, we have

$$k_1 v_1 = k_2 v_2 + k_3 v_3.$$

Eliminating v_1, v_2, v_3, between these four equations, we find that

$$k_1 x^{\frac{1}{2}} = k_2 (h - x)^{\frac{1}{2}} + k_3 (a - x)^{\frac{1}{2}},$$

whence x may be found by the solution of a quadratic equation. Having found x, we know v_1, v_2, v_3, by the first three equations.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 352.

8. $ABCD$, $GHIJ$, (fig. 41) are two vertical prismatic vessels, the rim of the aperture GJ in the upper vessel coinciding with the rim of the upper extremity of the lower vessel: the surface of the fluid in the compound vessel being supposed initially to coincide with AD, to determine the time in which it will subside to NP, owing to the efflux of the fluid through a small aperture in the base HI of the lower prism.

If $AE = a$, $BE = b$, $GH = a'$, $NH = x$; $K =$ the area of a horizontal section of the larger, and $K' =$ that of a horizontal section of the smaller prism, the required time will be equal to

$$\frac{1}{k} \cdot \left(\frac{2}{g}\right)^{\frac{1}{2}} \left(\frac{K}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} + \frac{K'}{a'^{\frac{1}{2}} + x^{\frac{1}{2}}}\right).$$

9. A vessel $ISTL$, (fig. 42), kept constantly full at the height TL, transmits fluid to the vertical prismatic vessel $AMNC$, by means of a thin horizontal tube TM: to determine the time in which the surface of the fluid in the prismatic vessel will rise from its initial position VK to any other position GE.

If a denote the distance of the surface VK, and x of the surface GE below the surface of the fluid in the vessel $ISTL$, A denoting the area of a horizontal section of the prismatic

vessel and M of a vertical section of the tube TM, the required time will be equal to

$$\left(\frac{2}{g}\right)^{\frac{1}{2}} \cdot \frac{A}{M} \cdot (a^{\frac{1}{2}} - x^{\frac{1}{2}}).$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 334.

10. The reservoir $ISTL$ (fig. 43), filled with fluid at first as far as IL, empties itself by the thin tube TM, which communicates with a second reservoir $AMNC$, containing initially fluid as far as DE, which discharges its contents through a small orifice at N. After a certain time the surfaces of the fluids in the two reservoirs arrive at QP and KV. To determine the relation between the vertical altitudes QH, KX, above the line of the tube, and the expression for the time of the motion.

Let $KX = x$, $QH = y$, $M =$ the area of the orifice at M, $N =$ the area of the orifice at N, $X =$ the area of the surface KV, $Y =$ the area of the surface QP. Then the values of x, y, t, will be connected together by the differential equations

$$\frac{Xdx}{M(y-x)^{\frac{1}{2}} - Nx^{\frac{1}{2}}} + \frac{Ydy}{M(y-x)^{\frac{1}{2}}} = 0,$$

$$dt = \frac{Xdx}{(2g)^{\frac{1}{2}} \cdot \{M(y-x)^{\frac{1}{2}} - Nx^{\frac{1}{2}}\}}.$$

The arbitrary constants introduced by the integration may be expressed in terms of the given initial altitudes ZX, OH, of the surfaces.

Bossut; *Traité d'Hydrodynamique*, tom. I. p. 337.

11. Two equal vertical prismatic vessels $ABCD$, $EFGH$, (fig. 44) which communicate together by a thin horizontal tube CG, are filled with fluid up to KL, MN: to determine the interval of time which will elapse before the surfaces of the fluid in the two vessels will be reduced to the same level.

If $A =$ the area of a horizontal section of either vessel, $M =$ the area of a vertical section of the tube, $KD = a$, $NH = b$, the required time will be equal to

$$\frac{A}{M} \cdot \left(\frac{a-b}{2g}\right)^{\frac{1}{2}}.$$

Bossut: *Traité d' Hydrodynamique*, tom. I. p. 339.

12. A vertical prismatic vessel, full of fluid, is divided into a number of compartments by a series of horizontal diaphragms, the compartments of the vessel communicating together by small orifices. Supposing the fluid in the upper compartment to be kept at its original altitude, and the fluids in the various compartments not to separate, to find the rate of efflux through the orifices.

If x_1, x_2, x_3, ... x_n, be the altitudes of the different compartments, k_1, k_2, k_3, ... k_n, the orifices, and h the altitude of the whole vessel, then, v_1, v_2, v_3, ... v_n, denoting the velocities at the orifices,

$$k_1 v_1 = k_2 v_2 = k_3 v_3 = \dots\dots = k_n v_n = \lambda,$$

where $$\lambda = \frac{(2gh)^{\frac{1}{2}}}{\left(\dfrac{1}{k_1^2} + \dfrac{1}{k_2^2} + \dfrac{1}{k_3^2} + \dots\dots + \dfrac{1}{k_n^2}\right)^{\frac{1}{2}}}.$$

Section VIII.

Efflux of Fluid through a small orifice of a vessel in motion.

1. A vessel $ABCD$ (fig. 45), containing fluid, being raised up vertically by a fine inextensible string $HNMP$, passing over two smooth pullies N and M, and having one end attached to a weight P: to determine the circumstances of the efflux at any time through a small orifice in the base.

Let f denote the acceleration of P, and therefore approximately of the whole moving system, and M the sum of the masses of the vessel and of the contained fluid, at any instant of time.

Then, by D'Alembert's Principle,

$$P(g - f) = M(g + f),$$

whence $$f = g\,\frac{P - M}{P + M}.$$

If therefore upon every molecule of the system we were to impress in an opposite direction, in addition to the force

of gravity, an acceleration equal to f or $g\,\dfrac{P-M}{P+M}$, we should reduce the vessel, as far as the pressure of the fluid within it is concerned, to the condition of a vessel suspended by a string NH tied to N. The accelerating force downwards upon each molecule of the fluid would accordingly be equal to

$$g+g\,\frac{P-M}{P+M}=\frac{2Pg}{P+M}.$$

The pressure therefore at any point of the base of the vessel, supposing x to be the depth and ρ the density of the fluid, will be equal to

$$\frac{2Pg\rho x}{P+M},$$

instead of its value in the case of a vessel at rest, viz. $g\rho x$. Hence, v denoting the velocity of efflux, we shall have,

$$v^2=\frac{4Pgx}{P+M}.$$

If k denote the area of the orifice, and Q the volume of the efflux in a time t,

$$dQ=kvdt=2k\left(\frac{Pg}{P+M}\right)^{\frac{1}{2}}x^{\frac{1}{2}}dt.$$

Let X represent the area of the surface of the fluid: then

$$dQ=-\,Xdx,$$

and $$M=A-\rho Q=A+\rho\int Xdx,$$

A representing the sum of the masses of the fluid and the vessel at the commencement of the motion.

From the last three equations we have

$$-\,Xdx=2k\left(\frac{Pg}{P+A+\rho\int Xdx}\right)^{\frac{1}{2}}x^{\frac{1}{2}}dt,$$

an equation from which, X being derived in terms of x from the form of the vessel, we may determine the relation between x and t.

If M be greater than P, the mass M will descend and P

ascend. In this case, as in the one we have been considering, the velocity of the efflux will be equal to the square root of $\frac{4Pgx}{P+M}$.

If, initially, $P = M$, then $f = 0$, $v^2 = 2gx$;

the vessel will therefore be at rest, at least for a moment, and the velocity of efflux will be the same as in a vessel without motion.

If $P = 0$, then $f = -g$, $v = 0$;

this will be the case of a vessel containing water and falling without constraint: in such a case it is evident that none of the fluid will escape, every molecule of the system falling at the same rate by the action of gravity.

If $P = \infty$, then $f = g$, $v = 2(gx)^{\frac{1}{2}}$.

Daniel Bernoulli: *Hydrodynam.*, Sect. XI. Art. 19.
D'Alembert: *Traité des Fluides*, p. 146.
Bossut: *Traité d'Hydrodynamique*, tom. I. p. 355.

2. A vessel $ABCD$ (fig. 46), containing fluid, is dragged along a smooth horizontal plane FQ by means of a weight P attached to the end of a string tied to the vessel at H and passing over a pully N: to find the form of equilibrium of the surface of the fluid and the pressure at any point of the fluid.

If M denote the sum of the masses of the fluid and the vessel, the acceleration of every particle of the system parallel to FQ will be equal to

$$\frac{Pg}{M+P}.$$

This acceleration of the molecules of the fluid is due to the resultant pressure of the vessel on the fluid arising from the acceleration of the vessel, the acceleration of the vessel being caused by the excess of the tension of the string above the resultant reaction of the fluid.

Suppose now an accelerating force $\frac{Pg}{M+P}$ to be impressed upon the whole system in a direction opposite to the motion: then the vessel may be regarded as at rest, and the fluid, instead

of experiencing the resultant pressure of the accelerated vessel, will be subject at every point to a horizontal force $\frac{Pg}{M+P}$, by which it will be caused to press against the vessel as if at rest.

Let x be the altitude of any molecule of the fluid above the base of the vessel, and y its distance from a plane fixed in the vessel at right angles to the direction of the motion.

Then, for the equilibrium of the fluid, there is

$$dp = \rho\left(\frac{Pg\,dy}{M+P} - g\,dx\right):$$

whence $$p = g\rho\left(\frac{Py}{M+P} - x\right) + c,$$

which determines the pressure at any point of the fluid, since the value of the constant may be found if the volume of the fluid be known.

Putting $p = 0$, we have, for the equation to the surface of the fluid,

$$0 = g\rho\left(\frac{Py}{M+P} - x\right) + c,$$

which defines a plane.

Cor. Let v represent the velocity with which the fluid would issue through a small aperture in the side of the vessel, and let x, y, be a, b, respectively at the aperture. Then

$$0 = g\rho\left(\frac{Pb}{M+P} - a\right) - \tfrac{1}{2}\rho v^2 + c',$$

and, x, y, being the coordinates of any point in the surface,

$$0 = g\rho\left(\frac{Py}{M+P} - x\right) + c',$$

or $$0 = -c + c';$$

hence $$\tfrac{1}{2}\rho v^2 = c + g\rho\left(\frac{Pb}{M+P} - a\right).$$

The quantity c in this equation will be some function of the volume of the fluid in the vessel at any instant of time or of the altitude of the intersection of any fixed vertical line with the

free surface above the base of the vessel. Thus the velocity of efflux may be determined for every position of the descending surface.

D'Alembert: *Traité des Fluides*, p. 146.
Bossut: *Traité d'Hydrodynamique*, tom. I. p. 359.

SECTION IX.

Efflux of Air from a vessel through a small orifice.

1. To determine the velocity of efflux when air escapes from a vessel into vacuum.

Suppose a vessel to be filled with air of its ordinary density at the surface of the earth. Then the pressure at any point will be equal to the weight of a column of homogeneous fluid of the same density as the air, the area of a transverse section of the column being unity, and its altitude about 29050 feet. Hence, a small orifice being made in the vessel so as to allow the air to escape into vacuum, the velocity of efflux, being that due to an altitude of 29050 feet, will be equal to

$$(2g \times 29050)^{\frac{1}{2}} = (64.4 \times 29050)^{\frac{1}{2}}$$
$$= 1367 \text{ feet nearly.}$$

The magnitude of this velocity of efflux will not be affected if the air in the vessel be supposed to have a different density from that of the atmosphere: for the velocity of any very small volume of effluent air depends upon the ratio of the pressure acting upon it to its density, and the pressure is proportional to the density.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 385.

2. To determine, under the circumstances of the preceding problem, the time in which the density of the air will decrease from one assigned amount to another.

Let C represent the capacity of the vessel, v the velocity of the efflux, k the area of the orifice; and let Q denote the mass of the contained air and ρ its density at any time t.

Then it is plain that $dQ = -\rho kvdt$,

and $$Q = C\rho:$$

hence $$C\frac{d\rho}{\rho} = -kv\,dt,$$

$$C\log\rho = \text{constant} - kvt.$$

Let ρ_1, ρ_2, be the values of ρ at the ends of the times t_1, t_2: then

$$C\log\frac{\rho_1}{\rho_2} = kv(t_2 - t_1),$$

or the interval between the density being ρ_1, ρ_2, is equal to

$$t_2 - t_1 = \frac{C}{kv}\log\frac{\rho_1}{\rho_2}.$$

If $\rho_2 = 0$, $t_2 - t_1 = \infty$, whence we conclude that the air would not all escape from the vessel in any finite time.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 386.

3. To determine the velocity with which air will escape from a vessel through a small orifice into circumambient air of indefinite extent, the density of which is not so great as that of the air in the vessel.

Let σ represent the density of the circumambient air and F its elastic force; ρ the initial density and accordingly $\frac{\rho}{\sigma}F$ the initial elastic force of the confined air; ρ' the density of the confined air after a certain time, and $\frac{\rho'}{\sigma}F$ its corresponding elastic force; M the small mass of air which issues initially in a certain small time with a velocity V; M' the small mass of air which issues in an equal portion of time some time afterwards, with a velocity V'.

The initial expulsive force will therefore be equal to

$$\frac{\rho}{\sigma}F - F = \frac{\rho - \sigma}{\sigma}F,$$

and the expulsive force after a time t will be equal to

$$\frac{\rho'}{\sigma}F - F = \frac{\rho' - \sigma}{\sigma}F.$$

Now the expulsive forces vary as the quantities of motion which they generate in a given time: hence

$$\frac{\rho-\sigma}{\sigma}F : \frac{\rho'-\sigma}{\sigma}F :: MV : M'V'.$$

But M, M', are equal to ρkV, $\rho' kV'$, respectively, k denoting the area of the orifice; hence

$$\rho - \sigma : \rho' - \sigma :: \rho V^2 : \rho' V'^2,$$

and therefore
$$V' = V\left\{\frac{\rho(\rho'-\sigma)}{\rho'(\rho-\sigma)}\right\}^{\frac{1}{2}}.$$

V is known from the relation

$$V^2 = 2g \cdot \frac{\rho-\sigma}{\rho} \cdot 29050.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 388.

4. Under the circumstances of the preceding problem, to find how long the air in the vessel will be decreasing in density from ρ to ρ'.

C denoting the capacity of the vessel and Q the mass of the contained air at any time t, we have

$$dQ = -\rho' kV' dt,$$

$$Cd\rho' = -k\rho' V\left\{\frac{\rho(\rho'-\sigma)}{\rho'(\rho-\sigma)}\right\}^{\frac{1}{2}} dt,$$

and therefore

$$dt = -\frac{C(\rho-\sigma)^{\frac{1}{2}}}{Vk\rho^{\frac{1}{2}}} \cdot \frac{d\rho'}{(\rho'^2-\sigma\rho')^{\frac{1}{2}}},$$

$$t = \text{constant} - \frac{C(\rho-\sigma)^{\frac{1}{2}}}{Vk\rho^{\frac{1}{2}}} \cdot \log\left\{\rho' - \tfrac{1}{2}\sigma + (\rho'^2 - \sigma\rho')^{\frac{1}{2}}\right\},$$

$$0 = \text{constant} - \frac{C(\rho-\sigma)^{\frac{1}{2}}}{Vk\rho^{\frac{1}{2}}} \cdot \log\left\{\rho - \tfrac{1}{2}\sigma + (\rho^2 - \sigma\rho)^{\frac{1}{2}}\right\},$$

and consequently

$$t = \frac{C(\rho-\sigma)^{\frac{1}{2}}}{Vk\rho^{\frac{1}{2}}} \log\frac{\rho - \frac{1}{2}\sigma + (\rho^2-\sigma\rho)^{\frac{1}{2}}}{\rho' - \frac{1}{2}\sigma + (\rho'^2-\sigma\rho')^{\frac{1}{2}}}.$$

Bossut: *Ib.* p. 390.

5. A vessel being supposed to contain air more rare than that of the atmosphere, to find the velocity with which the

surrounding air will flow into the vessel through a small orifice, and to determine the relation between the time and the density of the air in the vessel.

Let σ denote the density of the circumambient air, ρ the initial density of the air in the vessel, and ρ' its density at the end of any time t. Then, V denoting the initial velocity of the influx and V' its velocity at the time t, C the capacity of the vessel and k the area of the orifice,

$$V' = V\left(\frac{\sigma - \rho'}{\sigma - \rho}\right)^{\frac{1}{2}},$$

and
$$t = \frac{C(\sigma - \rho)^{\frac{1}{2}}}{\sigma k V}\{(\sigma - \rho)^{\frac{1}{2}} - (\sigma - \rho')^{\frac{1}{2}}\}.$$

Bossut: *Ib.*, pp. 391, 392.

6. Two closed vessels containing air of unequal condensation, to determine the velocity of the efflux of the air from one vessel into the other through a small orifice, and to find the equation between the time and the corresponding density of the air in either vessel.

Let ρ denote the initial density of the air in the vessel which contains the denser air, and ρ' its density at the end of a time t: let σ, σ', denote analogous quantities in regard to the other vessel. Then, A, B, denoting the capacities of the vessels of denser and rarer air respectively, k the area of the orifice, V the initial velocity of efflux and V' after a time t, we shall have

$$V' = V.\left[\frac{\rho\{B(\rho' - \sigma) - A(\rho - \rho')\}}{B\rho'(\rho - \sigma)}\right]^{\frac{1}{2}},$$

$$t = \frac{Am^{\frac{1}{2}}}{kVf^{\frac{1}{2}}}\log\left\{\frac{\rho - \dfrac{F\rho}{2f} + \left(\rho^2 - \dfrac{F\rho^2}{f}\right)^{\frac{1}{2}}}{\rho' - \dfrac{F\rho}{2f} + \left(\rho'^2 - \dfrac{F\rho\rho'}{f}\right)^{\frac{1}{2}}}\right\},$$

where $F = A\rho + B\sigma$, $f = (A + B)\rho$, $m = B(\rho - \sigma)$.

Bossut: *Ib.*, pp. 392, 393, 394.

CHAPTER II.

OSCILLATION OF FLUIDS IN PIPES.

THE general problem of the motion and pressure of fluids in pipes with either uniform or variable bores is discussed at length, with many interesting applications, by Euler, in a Memoir in the *Novi Commentarii Academiæ Scientiarum Petropolitanæ*, tom. XV., *pro anno* 1770. This memoir consists of five chapters: — Caput I. *De principiis motus linearis fluidorum.* Cap. II. *De motu aquæ in tubis æqualiter ubique amplis.* Cap. III. *De motu aquæ in tubis inæqualiter amplis.* Cap. IV. *De elevatione aquæ antliarum ope.* Cap. V. *De motu aquæ per tubos diverso caloris gradu infectos.* The reader may consult also a memoir by M. Coriolis, entitled, "*Sur une manière simple de calculer la pression produite contre les parois d'un canal dans lequel se meut un fluide incompressible:* in Liouville's *Journal de Mathématiques, année* 1837; p. 130.

1. An incompressible fluid oscillates in a siphon $ABCD$ of uniform bore, (fig. 47) which consists of two vertical branches and one horizontal: to ascertain the period of an oscillation.

Let x, y, be the altitudes of the two free surfaces of the column above the horizontal portion of the siphon. Let c represent the whole length of the column of fluid, and a the length of the horizontal branch of the siphon. Then, k denoting the area of a section of the siphon and ρ the density of the fluid, the mass of fluid in motion will be equal to $k\rho c$, and the force producing motion, being the difference of the weights of the two vertical portions of the fluid, will be equal to

$$g\rho k\,(y - x):$$

hence $$k\rho c\,\frac{d^2x}{dt^2} = g\rho k\,(y - x), \qquad \frac{d^2x}{dt^2} + \frac{g}{c}\,(x - y) = 0.$$

But, a denoting the length of the horizontal branch of the siphon,

$$x + y + a = c:$$

hence

$$\frac{d^2x}{dt^2} + \frac{g}{c}(2x + a - c) = 0,$$

$$\frac{d^2}{dt^2}\{x + \tfrac{1}{2}(a - c)\} + \frac{2g}{c}\{x + \tfrac{1}{2}(a - c)\} = 0,$$

which shews that the period of an oscillation is equal to $\pi\left(\frac{c}{2g}\right)^{\frac{1}{2}}$, and is therefore the same as that of an oscillation of a perfect pendulum the length of which is half that of the column of fluid.

Newton compares the undulatory motion of the waves of an indefinite mass of fluid to the oscillation of the fluid in a siphon.

Newton: *Principia*, lib. II. Sect. vii. Prop. 44, 45, 46.
Euler: *Novi Comment. Petropolit.*, an. 1770, p. 251.
Bossut: *Traité d'Hydrodynamique*, tom. I. p. 363.

2. An incompressible fluid oscillates under the action of gravity in a smooth continuous tube of variable bore: to determine the motion of the fluid.

We shall conceive the fluid to be divided into an infinite number of parallel sections, each of them pierced perpendicularly by the axis of the tube. We shall also suppose the motion of the particles in each slice to move at right angles to either face of the slice, in accordance with the hypothesis of parallel sections.

Let L (fig. 48) be the area of any section of the fluid, L' and L'' being the values of L at the extremities of the column of fluid. Let M be the area of a section of the fluid at any other point. Let u, v, be the velocities of the fluid at L, M. Let s, s', s'', be the lengths of the axis of the tube reckoned from a fixed point A in it to the sections M, L', L'', respectively, and let z be the depth of the point of the axis at M below the horizontal plane through A.

Then, by the equation of fluid motion, $g\,\dfrac{dz}{ds}$ being the component of gravity at M along the axis of the tube at that point,

$$\frac{dv}{dt}+v\,\frac{dv}{ds}=g\,\frac{dz}{ds}-\frac{1}{\rho}\,\frac{dp}{ds},$$

$$\int\frac{dv}{dt}\,ds+\tfrac{1}{2}\,v^2=gz-\frac{1}{\rho}\,p \;\dots\dots\dots\dots\dots\; (1).$$

Since the fluid is supposed to be incompressible and invariable in volume, it follows that the same quantity of fluid must pass through each section of the tube, which is occupied by fluid, in an assigned time: hence

$$Lu=Mv\dots\dots\dots\dots\dots\dots\dots\;(2).$$

Differentiating the equation (2) with regard to t, and observing that L and M, being fixed sections, do not vary with the time, we have

$$L\,\frac{du}{dt}=M\,\frac{dv}{dt},$$

and therefore, from (1), observing that L and $\dfrac{du}{dt}$ depend only upon t and the position of L, and not upon the position of M, that is, upon the magnitude of s, we get

$$L\,\frac{du}{dt}\int\frac{ds}{M}+\tfrac{1}{2}\,\frac{L^2}{M^2}\,u^2=gz-\frac{1}{\rho}\,p\dots\dots\dots\dots\;(3).$$

Supposing the pressure to be Π, a constant quantity, at both ends of the oscillating column, we have, z', z'', being the values of z at L', L'',

$$L\,\frac{du}{dt}\int_{s'}^{s''}\frac{ds}{M}+\tfrac{1}{2}\,L^2\left(\frac{1}{L''^2}-\frac{1}{L'^2}\right)u^2=g\,(z''-z')\dots\dots\;(4).$$

Again, from (2), we have

$$L'u'=Lu=L''u'',$$

or,
$$L'\,\frac{ds'}{dt}=Lu=L''\,\frac{ds''}{dt}\;\dots\dots\dots\dots\dots\;(5).$$

From the two equations (5) there is

$$L''ds''-L'ds'=0,$$

which is equivalent to

$$\int_{s'}^{s''} M ds = C \ldots\ldots\ldots\ldots\ldots \text{(6)},$$

C being a constant quantity, representing, as appears from the form of the left-hand member of the equation, the volume of the oscillating fluid.

From (6) then we know s'' in terms of s', and z', z'', which are given functions of s', s'', are therefore known in terms of s'. Substituting for s'', z', z'', in terms of s' in the equation (4), we have an equation in s', u, t, and from this final equation, combined with

$$L' \frac{ds'}{dt} = Lu \ldots\ldots\ldots\ldots\ldots\ldots \text{(7)},$$

we may find s' and u in terms of t. The two arbitrary constants introduced in the integration of these two equations may be determined when the initial values of s' and $\frac{ds'}{dt}$ are assigned.

The equation (4) may be obtained somewhat elegantly by the direct application of D'Alembert's Principle. Conceive the indefinitely small slices of the fluid to be taken all equal and to have each of them a mass m.

Then, $\frac{Dv}{dt}$ denoting the effective force and $g \frac{dz}{ds}$ the component of the impressed force in the direction in which the slice at M moves, we have, by D'Alembert's Principle combined with that of Virtual Velocities,

$$\int \left(g \frac{dz}{ds} - \frac{Dv}{dt} \right) m ds = 0,$$

the symbol of integration being supposed to extend from one end of the fluid to the other. Since m is the same for all the slices, this equation becomes

$$\int_{s'}^{s''} \left(g \frac{dz}{ds} - \frac{Dv}{dt} \right) ds = 0,$$

whence
$$g(z''-z')=\int_{s'}^{s''}\frac{Dv}{dt}\,ds$$
$$=\int_{s'}^{s''}\left(\frac{dv}{dt}+\frac{dv}{ds}\frac{ds}{dt}\right)ds$$
$$=\int_{s'}^{s''}\left(\frac{dv}{dt}+v\frac{dv}{ds}\right)ds$$
$$=\tfrac{1}{2}(u''^2-u'^2)+\int_{s'}^{s''}\frac{dv}{dt}\,ds.$$

But, as we have proved in the former solution,
$$L\frac{du}{dt}=M\frac{dv}{dt},$$
and therefore
$$g(z''-z')=L\frac{du}{dt}\int_{s'}^{s''}\frac{ds}{M}+\tfrac{1}{2}(u''^2-u'^2)$$
$$=L\frac{du}{dt}\int_{s'}^{s''}\frac{ds}{M}+\tfrac{1}{2}L^2\left(\frac{1}{L''^2}-\frac{1}{L'^2}\right)u^2.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 365.

3. Fluid oscillates in a cylindrical tube, the axis of which is a circle in a vertical plane: to determine the motion.

Let C (fig. 49) be the centre of the circle, CA a horizontal radius, $L'L''$ the fluid in any position. Join CL', CL''. Let $\angle ACL'=\theta$, $\angle L'CL''=\alpha$, $r=$ the radius.

Then, retaining the notation of the preceding problem, and observing that all the sections are equal, we have from the equation (4),
$$\frac{du}{dt}(s''-s')=g(z''-z')\ \ldots\ldots\ldots\ldots(8),$$
and, from the equation (6),
$$M(s''-s')=C,$$
or, c denoting the length of the column of fluid,
$$s''-s'=c\ \ldots\ldots\ldots\ldots\ldots\ldots(9).$$
Also, from (7),
$$\frac{ds'}{dt}=u\ \ldots\ldots\ldots\ldots\ldots\ldots(10).$$
From (8), (9), (10), there is
$$c\frac{d^2s'}{dt^2}=g(z''-z')=gr\left(\sin\frac{s''}{r}-\sin\frac{s'}{r}\right),$$

or $$c\,\frac{d^2\theta}{dt^2} = g\,\{\sin(\theta + \alpha) - \sin\theta\}$$

$$= 2g\sin\frac{\alpha}{2}\cos\left(\theta + \frac{\alpha}{2}\right) \text{..............} (11),$$

whence $$c\,\frac{d\theta^2}{dt^2} = 4g\sin\frac{\alpha}{2}\sin\left(\theta + \frac{\alpha}{2}\right) + \text{constant}:$$

let $\dfrac{d\theta}{dt} = 0$, when $\theta = \beta$: then

$$0 = 4g\sin\frac{\alpha}{2}\sin\left(\beta + \frac{\alpha}{2}\right) + \text{constant};$$

$$c\,\frac{d\theta^2}{dt^2} = 4g\sin\frac{\alpha}{2}\left\{\sin\left(\theta + \frac{\alpha}{2}\right) - \sin\left(\beta + \frac{\alpha}{2}\right)\right\},$$

$$dt = \frac{c^{\frac{1}{2}}}{2\left(g\sin\frac{\alpha}{2}\right)^{\frac{1}{2}}} \cdot \frac{d\theta}{\left\{\sin\left(\theta + \frac{\alpha}{2}\right) - \sin\left(\beta + \frac{\alpha}{2}\right)\right\}^{\frac{1}{2}}}.$$

In order to simplify results, let us suppose that the oscillations of the fluid are small. Let $\theta = \dfrac{\pi}{2} - \dfrac{\alpha}{2} + \zeta$, $\dfrac{\pi}{2} - \dfrac{\alpha}{2}$ being the value of θ when the fluid is in equilibrium. Then, from (11),

$$c\,\frac{d^2\zeta}{dt^2} = 2g\sin\frac{\alpha}{2}\cos\left(\frac{\pi}{2} + \zeta\right)$$

$$= -\,2g\sin\frac{\alpha}{2}\,.\,\zeta \text{ nearly}:$$

the integral of this equation is

$$\zeta = A\sin\left\{\left(\frac{2g}{c}\sin\frac{\alpha}{2}\right)^{\frac{1}{2}} t + \varepsilon\right\},$$

A and ε being arbitrary constants: whence

$$\theta + \tfrac{1}{2}(\alpha - \pi) = A\sin\left\{\left(\frac{2g}{c}\sin\frac{\alpha}{2}\right)^{\frac{1}{2}} t + \epsilon\right\}.$$

When $\dfrac{d\theta}{dt} = 0$, let $\theta = \beta$ and $t = 0$: then

$$\beta + \tfrac{1}{2}(\alpha - \pi) = A\sin\varepsilon,$$

and $$0 = A \cos \epsilon:$$

hence $$\theta = \tfrac{1}{2}(\pi - \alpha) + \{\beta - \tfrac{1}{2}(\pi - \alpha)\} \cos \left\{\left(\frac{2g}{c} \sin \frac{\alpha}{2}\right)^{\frac{1}{2}} t\right\}.$$

Bossut: *Traité d'Hydrodynamique,* tom. I. p. 368.

4. Fluid oscillates in a siphon of uniform bore which consists of two branches inclined at given angles to the vertical connected by a horizontal branch: to find the length of a perfect pendulum which shall oscillate isochronously with the fluid.

If c represent the whole length of the column of fluid, and α, β, the angles at which its extreme branches are inclined to the vertical, the length of the perfect pendulum will be equal to

$$\frac{c}{\cos \alpha + \cos \beta}.$$

John Bernoulli: *Opera,* tom. III. p. 125.
Bossut: *Traité d'Hydrodynamique,* tom. I. p. 371.

5. The fluid in the tubes of a differential thermometer having the same section throughout, makes small oscillations in a vertical plane: to determine the length of the isochronous simple pendulum.

Let k represent the area of a section of the tube, v the volume of air in the bulb and tube together on each side when the fluid is in a position of equilibrium, h the height of a column of the contained fluid which would press with a force equal to that of the air when the fluid is in its position of rest, and l the whole length of the oscillating column. Then the length of the isochronous pendulum will be equal to

$$\frac{lv}{2(hk + v)}.$$

6. A thin tube of uniform bore consists of two straight branches AB and BC (fig. 50) inclined at given angles to the horizon: the branch BC is free from air and closed at C, while BA is open to the atmosphere at A: in this tube there moves a filament MBN of fluid of given length: to determine its motion.

Let x represent the length BM, l the whole length of the filament; ϵ, ζ, the inclinations of AB, CB, to the horizon; k the height of a column of the fluid which would exert a pressure equal to that of the atmosphere. Then the motion of the fluid will be defined by the differential equation

$$l\frac{d^2x}{dt^2}+g\{k-l\sin\zeta+(\sin\epsilon+\sin\zeta)\,x\}=0,$$

which is easily integrated. The arbitrary constants introduced by the integration are to be determined by knowing the position and velocity of the filament at any instant of time.

Euler: *Comment. Acad. Petrop.* tom. xv. p. 254.

CHAPTER III.

RESISTANCES.

THE earliest problems of the resistance of a fluid, on the surface of a solid moving through it, were solved by Newton,* who ascertained the magnitude of the resistance on a globe and on a cylinder moving in the direction of its axis, and enunciated the fundamental property of the *Solid of Least Resistance.* The theory of resistances laid down by Newton received afterwards extensive applications in the hands of James Bernoulli, who has given the results of his calculations for various forms of surfaces in the *Acta Eruditorum* for 1693. John Bernoulli has also treated on this subject in his *Nouvelle Theorie de la Manœuvre,* and Herman has devoted to it the twelfth chapter of the second book of his *Phoronomia.*

If A be the area of a plane directly opposed to a stream of fluid, of which the density is ρ, moving with the velocity v; then, according to the ordinary theory, the resistance on the plane will be equal to $\frac{1}{2}\rho v^2 A$. This formula, which forms the basis of the ordinary theory of resistances, was first deduced from the equations of fluid motion by John Bernoulli,† the law of resistance which it expresses having been already adopted by Newton, James Bernoulli, and others.

An interesting application of the theory of Resistances has been made by Euler, in a memoir on the maximum efficiency of Wind Mills, in the *Commentarii Academiæ Petropolitanæ,* tom. IV. p. 41, entitled *De Constructione Aptissima Molarum Alatarum.*

In the following problems, wherever the contrary is not stated, the resistance will be supposed to take place according to the law of the square of the velocity.

* Principia: lib. II. Sec. 7.
† Comment. Acad. Petrop. 1737, p. 37.

1. An isosceles triangular lamina being exposed to the action of a stream, the direction of which is parallel to the perpendicular distance CD (fig. 51) of the vertex C of one face of the lamina from the base AB of this face: to compare the resultant pressure of the fluid on the lamina as its vertex or its base is opposed to the stream.

Let v be the velocity of the stream, 2α the angle at the vertex of either face of the lamina, l the length of AC or BC, s the distance of any point P in AC from C, and τ the thickness of the lamina. Let P denote the pressure on the lamina, when C, and Q, when AB meets the current.

Then, under the supposition that C meets the stream, the component of the velocity at P, at right angles to AC, is $v \sin \alpha$, and therefore the normal pressure, on an elemental rectangle τds at P, will be equal to

$$\tfrac{1}{2}\rho\tau\, ds\,.\,(v \sin \alpha)^2;$$

and the component of this pressure, parallel to CD, will be equal to

$$\tfrac{1}{2}\rho\tau\, ds\,.\,(v \sin \alpha)^2 . \sin \alpha = \tfrac{1}{2}\rho\tau v^2 \sin^3 \alpha\, ds.$$

Hence $$P = 2\int_0^l \tfrac{1}{2}\rho\tau v^2 \sin^3 \alpha\, ds = \rho\tau l v^2 \sin^3 \alpha.$$

When AB meets the stream, each of its elements being acted on perpendicularly by the fluid with a velocity v, we shall have

$$Q = \tfrac{1}{2}\rho\tau(2l \sin \alpha)\, v^2$$
$$= \rho\tau l v^2 \sin \alpha.$$

Hence $$P : Q :: \sin^2 \alpha : 1,$$

or, if a denote the length of the base,

$$P : Q :: a^2 : 4l^2.$$

James Bernoulli: *Acta Eruditorum, Lips.* 1693, p. 252.
Herman: *Phoronomia*, p. 245.
Bossut: *Traité d' Hydrodynamique*, tom. I. p. 431.

2. A lamina in the form of a semi-ellipse, bounded by the minor axis, moves through a fluid, first with its vertex and next with its base foremost, in the direction of its axis: to compare the resultant resistances in the two cases.

Let v be the velocity of the stream, a the length of the semi-axis major AC (fig. 52), $2b$ the length of the axis minor BB', of either face of the lamina. Let CAx, CBy, be taken as the axes of coordinates, and let $CM = x$, $PM = y$. Let $Pp = ds$, p being any point of the curve AB indefinitely near to P. Then, τ denoting the thickness of the lamina and ρ the density of the fluid, the normal resistance on the portion Pp of the lamina, the vertex A being supposed to move foremost, will be equal to

$$\tfrac{1}{2}\rho\tau v^2 \frac{dy^2}{ds^2}\, ds,$$

and the component of this, parallel to AC, will be equal to

$$\tfrac{1}{2}\rho\tau v^2 \frac{dy^3}{ds^3}\, ds.$$

Hence, P denoting the resultant resistance on the lamina, parallel to AC,

$$P = \tfrac{1}{2}\rho\tau v^2 \int \frac{dy^3}{ds^3}\, ds = \tfrac{1}{2}\rho\tau v^2 \int \frac{dy}{1 + \dfrac{dx^2}{dy^2}}.$$

But, from the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we may get

$$\frac{dx^2}{dy^2} = \frac{a^2}{b^2} \cdot \frac{y^2}{b^2 - y^2};$$

hence, integrating from $-y$ to $+y$, we have

$$\begin{aligned} P &= \tfrac{1}{2}\rho\tau v^2 b^2 \int \frac{(b^2 - y^2)\, dy}{b^4 + (a^2 - b^2)\, y^2} \\ &= \tfrac{1}{2}\rho\tau v^2 \frac{b^2}{a^2 - b^2} \int \left\{ \frac{a^2 b^2}{b^4 + (a^2 - b^2)\, y^2} - 1 \right\} dy \\ &= \tfrac{1}{2}\rho\tau v^2 . \frac{2b^2}{a^2 - b^2} \left\{ \frac{a^2}{(a^2 - b^2)^{\frac{1}{2}}} \tan^{-1} \frac{(a^2 - b^2)^{\frac{1}{2}}\, y}{b^2} - y \right\}, \end{aligned}$$

or, b being the value of y in the limit,

$$P = \tfrac{1}{2}\rho\tau v^2 . \frac{2b^2}{a^2 - b^2} \cdot \left\{ \frac{a^2}{(a^2 - b^2)^{\frac{1}{2}}} \tan^{-1} \frac{(a^2 - b^2)^{\frac{1}{2}}}{b} - b \right\}.$$

But, Q denoting the resistance when the motion of the lamina takes place in the opposite direction,

$$Q = \tfrac{1}{2}\rho\tau v^2 . 2b:$$

hence $$\frac{P}{Q} = \frac{b}{a^2 - b^2}\left\{\frac{a^2}{(a^2 - b^2)^{\frac{1}{2}}} . \tan^{-1}\frac{(a^2 - b^2)^{\frac{1}{2}}}{b} - b\right\}.$$

James Bernoulli: *Acta Eruditorum, Lips.* 1693, p. 253, Art. 5.

Herman: *Phoronomia*, p. 246.

3. A semicircular lamina is placed in a stream flowing in a direction parallel to the axis of the lamina: to compare the resultant stress experienced by the lamina, when its vertex meets the stream, with that which it experiences when its base meets the stream.

Let C (fig. 53) be the middle point of the rectilinear boundary of either face of the lamina, P any point in the curvilinear circumference of this face, and CA its axis. Let $AC = a$, $\angle ACP = \theta$, arc $AP = s$, τ = the thickness of the lamina. Let P, Q, denote the resultant pressures of the fluid on the lamina, accordingly as A or C meets the stream.

Then, $v \cos\theta$ being the normal component of the velocity with which the fluid impinges upon the lamina at P, the component of the pressure upon an elemental rectangle τds of the edge of the lamina, parallel to AC, will be equal to

$$\tfrac{1}{2}\rho . \tau ds . (v\cos\theta)^2 . \cos\theta = \tfrac{1}{2}\rho\tau a v^2 \cos^3\theta\, d\theta.$$

Hence, P being equal to twice the component of the pressure upon the quadrant AB, parallel to AC, we have

$$P = \rho\tau a v^2 \int_0^{\frac{1}{2}\pi} \cos^3\theta d\theta = \rho\tau a v^2 \int_0^1 (1 - \sin^2\theta)\, d\sin\theta$$

$$= \rho\tau a v^2 (1 - \tfrac{1}{3}) = \tfrac{2}{3}\rho\tau a v^2.$$

The pressure Q upon the base of the lamina will be equal to

$$\tfrac{1}{2}\rho . 2\tau a . v^2 = \rho\tau a v^2.$$

Hence $$P : Q :: 2 : 3.$$

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 433—437.

4. To find the centre of resistance on a semicircular plate, revolving about its diameter in a medium, the resistance of which varies as the square of the velocity.

Let ω denote the angular velocity of revolution, a the radius of the circle, x the distance of the centre of resistance from the axis of revolution. Then, r being the distance of an elemental area $rd\theta dr$ from the centre of the circle, and θ the inclination of r to the axis of the semicircle, we shall have

$$x\iint rd\theta dr \,.\, \tfrac{1}{2}\rho\,(\omega r\cos\theta)^2 = \iint rd\theta dr \,.\, r\cos\theta \,.\, \tfrac{1}{2}\rho\,(\omega r\cos\theta)^2,$$

or
$$x\iint r^3\cos^2\theta d\theta dr = \iint r^4\cos^3\theta d\theta dr,$$

where the integrations are to be performed first from $r = 0$ to $r = a$, and then from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

Hence
$$\tfrac{1}{4}x\int\cos^2\theta d\theta = \tfrac{1}{5}a\int\cos^3\theta d\theta,$$

$$\tfrac{1}{8}x\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{5}a\int(1-\sin^2\theta)\,d\sin\theta,$$

$$\tfrac{1}{8}x\,(\theta + \tfrac{1}{2}\sin 2\theta) = \tfrac{1}{5}a\,(\sin\theta - \tfrac{1}{3}\sin^3\theta),$$

$$\frac{\pi x}{16} = \frac{2a}{15},$$

or
$$x = \frac{32a}{15\pi}.$$

5. An indefinitely thin rectangular plank AB, moveable about one of its ends A, which is fixed horizontally, is plunged vertically up to A into a current flowing at right angles to the face of the plank, and then abandoned to the stream: to determine that point C of the plank of which, *ipso motûs initio*, the velocity is the same as that of the fluid.

Let $AB = a$, $AC = x$ (fig. 54); let y denote the distance of any point P in the plank from the end A; let b denote the breadth of the plank, and v the velocity of the fluid and of the point C.

Then the velocity of P will be $\dfrac{vy}{x}$, and the difference between

this and the velocity of the fluid will be

$$v \sim \frac{vy}{x} = \frac{v}{x}(x \sim y).$$

Now the action of the fluid on AC will tend to accelerate and that on BC to retard the rotation of the plank about A: and, since the moment of inertia of the plank is supposed to be inconsiderable, the moments of these two actions of the fluid must be equal: hence

$$\tfrac{1}{2}\rho\int_0^x \frac{v}{x}(x \sim y)^2 . bydy = \tfrac{1}{2}\rho\int_x^a \frac{v}{x}(x \sim y)^2 . bydy,$$

or, observing that $(x \sim y)^2$ is equal to $(x - y)^2$,

$$\int_0^x (x-y)^2\, ydy = \int_x^a (x-y)^2\, ydy = \int_0^a (x-y)^2\, ydy - \int_0^x (x-y)^2\, ydy,$$

whence $$2\int_0^x (x-y)^2\, ydy = \int_0^a (x-y)^2\, ydy,$$

$$2\int_0^x (x^2 - 2xy + y^2)\, ydy = \int_0^a (x^2 - 2xy + y^2)\, ydy,$$

$$2\left(\tfrac{1}{2}x^4 - \tfrac{2}{3}x^4 + \tfrac{1}{4}x^4\right) = \tfrac{1}{2}a^2x^2 - \tfrac{2}{3}a^3x + \tfrac{1}{4}a^4,$$

$$x^4 - 3a^2x^2 + 4a^3x - \tfrac{3}{2}a^4 = 0,$$

from which the required value of x is to be obtained.

Approximately $$x = \frac{23}{36}a.$$

Ducrest: *Essais sur les Machines Hydrauliques*, p. 267, Paris, 1777.

6. A thin rod is suspended by its middle point to the end of a fine string, which, being twisted through an angle α, tends to regain its former position: to find the angle θ which the rod makes with its position of rest at the end of any time t; T being the time of an oscillation, and the resistance on a small length dr of the rod being supposed to vary as vdr, where v is the velocity of dr.

Since the force of torsion varies as θ, we shall have for the motion of the rod, r being the distance of dr from the string,

λ the area of a transverse section of the rod, and c, k, constant quantities,

$$\int \lambda r^2 dr \frac{d^2\theta}{dt^2} = -c\theta - \int \lambda kr \frac{d\theta}{dt} dr \cdot r,$$

$$\frac{d^2\theta}{dt^2} = -\beta\theta - k\frac{d\theta}{dt},$$

where β represents some constant quantity.

Assume $\theta = e^{mt}$: then

$$m^2 + km + \beta = 0,$$

$$m = -\tfrac{1}{2}k \pm (-1)^{\frac{1}{2}} \cdot \tfrac{1}{2}(4\beta - k^2)^{\frac{1}{2}}:$$

hence, putting $\mu = \frac{1}{2}(4\beta - k^2)^{\frac{1}{2}}$, we have for the complete integral

$$\theta = e^{-\frac{1}{2}kt} \cdot C \sin(\mu t + C'),$$

where C, C', are arbitrary constants.

Let $\theta = 0$: then

$$\mu t + C' = 0, \quad \text{or } \mu t + C' = \pi,$$

or $-\dfrac{C'}{\mu}$, $\dfrac{\pi - C'}{\mu}$, are successive values of t corresponding to the zero value of θ: hence, T denoting the time of an oscillation,

$$\mathrm{T} = \frac{\pi}{\mu}.$$

Hence $$\theta = e^{-\frac{1}{2}kt}\left(A \cos\frac{\pi t}{\mathrm{T}} + B \sin\frac{\pi t}{\mathrm{T}}\right),$$

A, B, being arbitrary constants.

If $\dfrac{d\theta}{dt} = 0$, when $\theta = a$ and $t = 0$, it may be easily ascertained that

$$A = a, \quad B = \frac{ka\mathrm{T}}{2\pi}:$$

hence $$\theta = e^{-\frac{1}{2}kt} \cdot \left(a \cos\frac{\pi t}{T} + \frac{kaT}{2\pi} \sin\frac{\pi t}{T}\right).$$

7. A plane is moveable at right angles to a stream, the direction of which makes an angle a with the normal, and the force required to retain it at a constant velocity v in one direction is four times that required to retain it at an equal velocity in the opposite direction. Assuming the common theory of resistances, to find the velocity of the stream.

First we will suppose that the requisite forces both act in the same direction. Let v' be the velocity of the stream. Let OA (fig. 55) be one of the directions of the board's motion, and AO the opposite direction. Then, in the former motion, the pressure of the fluid on the board is equal to

$$k\,(v' \cos \alpha + v \sin \alpha)^2,$$

k being a constant quantity: and therefore the pressure of the fluid parallel to AO is equal to

$$k\,(v' \cos \alpha + v \sin \alpha)^2 \sin \alpha.$$

For the opposite motion of the board, the pressure of the fluid on the board parallel to AO, putting $-v$ for v, is equal to

$$k\,(v' \cos \alpha - v \sin \alpha)^2 . \sin \alpha.$$

Since these two pressures must be equal to the pressures applied to the board in the two cases to retain it at the proposed velocity, it follows that

$$k\,(v' \cos \alpha + v \sin \alpha)^2 \sin \alpha = 4k\,(v' \cos \alpha - v \sin \alpha)^2 \sin \alpha,$$

$$v' \cos \alpha + v \sin \alpha = 2v' \cos \alpha - 2v \sin \alpha,$$

whence $$v' = 3v \tan \alpha.$$

Secondly, suppose the requisite forces to act in opposite directions: then, $v \sin \alpha$ being greater than $v' \cos \alpha$,

$$k\,(v' \cos \alpha + v \sin \alpha)^2 \sin \alpha = 4k\,(v \sin \alpha - v' \cos \alpha)^2 \sin \alpha,$$

$$v' \cos \alpha + v \sin \alpha = 2\,(v \sin \alpha - v' \cos \alpha),$$

$$v' = \tfrac{1}{3}\, v \tan \alpha.$$

8. A thin rectangular plank is revolving about one end, which is horizontal, with a given angular velocity: its lower end is at a given depth in a horizontal current. To find the inclination of the plank to the vertical, that the moment of the action of the fluid on the immersed portion about the fixed end may be the greatest possible, the direction of the horizontal component of the velocity of each point of the plank being supposed to be the same as that of the current.

Let O, A, (fig. 56), be the middle points of the two ends of the plank. Draw AF horizontally to meet the vertical line through O in the point F: let E be the point in which OF

intersects the surface of the fluid. Let $\angle AOF = \alpha$, $OE = a$, $OF = b$. Take any point P in OA and draw PQ horizontally to meet OF in Q: let $OQ = r$, $OP = r'$. Let ω = the angular velocity of the plank, u = the velocity of the current, and c = the breadth of the plank.

The relative normal velocity with which the current is incident upon the element cdr' of the plank is equal to

$$u \cos \alpha - r'\omega,$$

and therefore the pressure on this element is equal to

$$\tfrac{1}{2}\rho c dr' (u \cos \alpha - r'\omega)^2:$$

hence the moment, about O, of the whole action of the fluid on the plank will be equal to

$$\tfrac{1}{2}\rho c \int r' (u \cos \alpha - r'\omega)^2 \, dr'$$

$$= \tfrac{1}{2}\rho c \int r \left(u - r \frac{\omega}{\cos^2 \alpha}\right)^2 dr,$$

the integration being performed from $r = a$ to $r = b$.

Since $\left(u - r \dfrac{\omega}{\cos^2 \alpha}\right)^2$ is manifestly less than $(u - r\omega)^2$, it follows that the value of the moment of the fluid's action will be greatest when $\alpha = 0$.

9. A thin rectangular plank moveable about one end, which is fixed horizontally to the axis of an axle, is partially immersed in a current, and raises a weight attached to a fine string which passes over a fixed pully of inconsiderable inertia and is wound round the axle. To determine the differential equation for the motion of the plank.

Let O, A (fig. 57), be the middle points of the two ends of the plank, E the point in which the string touches the axle OE, F the fixed pully, W the weight, and Q the intersection of OA with the surface of the fluid. Let OCK be an indefinite vertical line through O, intersecting the surface of the fluid in C.

Let m = the mass of W, T = the tension of the string, r = the radius of the axle, b = the breadth of the plank, $OA = a$, $OC = c$, $\theta = \angle AOK$, at any time t, Mk^2 = the moment of inertia of the plank and axle about the axis of revolution, u = the velocity of the

current, s = the distance of any point P in AQ from the point O, x = the distance of W below any fixed horizontal plane.

Then, for the motion of the weight,

$$m\frac{d^2x}{dt^2} = mg - \mathrm{T},$$

or, since $dx = r\,d\theta$,

$$mr\frac{d^2\theta}{dt^2} = mg - \mathrm{T} \ldots\ldots\ldots\ldots\ldots (1).$$

For the motion of the plank, QC being the direction of u and $\frac{d\theta}{dt}$ being negative,

$$Mk^2\frac{d^2\theta}{dt^2} = \mathrm{T}r - \tfrac{1}{2}\rho b\int_{\frac{c}{\cos\theta}}^{a}\left(u\cos\theta + s\frac{d\theta}{dt}\right)^2 s\,ds \ldots. (2).$$

From the equations (1) and (2), we have

$$(Mk^2 + mr^2)\frac{d^2\theta}{dt^2} = mgr - \tfrac{1}{2}\rho b\int_{\frac{c}{\cos\theta}}^{a}\left(u\cos\theta + s\frac{d\theta}{dt}\right)^2 sds$$

$$= mgr - \tfrac{1}{2}\rho b\int_{\frac{c}{\cos\theta}}^{a}\left(u^2\cos^2\theta + 2us\cos\theta\frac{d\theta}{dt} + s^2\frac{d\theta^2}{dt^2}\right)s\,ds$$

$$= mgr - \tfrac{1}{2}\rho b\left\{\tfrac{1}{2}u^2\cos^2\theta\left(a^2 - \frac{c^2}{\cos^2\theta}\right) + \tfrac{2}{3}u\cos\theta\frac{d\theta}{dt}\left(a^3 - \frac{c^3}{\cos^3\theta}\right)\right.$$

$$\left. + \tfrac{1}{4}\left(a^4 - \frac{c^4}{\cos^4\theta}\right)\frac{d\theta^2}{dt^2}\right\} \ldots\ldots (3),$$

which is the differential equation to the motion of the plank.

COR. Suppose that a considerable number of such planks, forming float-boards of a water-wheel, are fixed to the axis O. Suppose that OC is so large compared with the immersed length of any of the float-boards, that θ may be always regarded as zero, and suppose that the wheel has arrived at its permanent velocity of rotation, slight inequalities of motion being disregarded. We may conceive the floats to be so adjusted that one only may be immersed at a time, or that, more than one being partially immersed, the effect may be the same as one totally immersed. Then from (3), $\frac{d^2\theta}{dt^2}$ being zero, we have,

putting $-\omega$ for $\dfrac{d\theta}{dt}$,

$$mgr = \tfrac{1}{2}\rho b\left\{\tfrac{1}{2}u^2(a^2-c^2) - \tfrac{2}{3}u(a^3-c^3)\omega + \tfrac{1}{4}(a^4-c^4)\omega^2\right\}.$$

If v = the velocity of the weight, and $mg = W$, we have,

$$Wv = \tfrac{1}{2}\rho b\left\{\tfrac{1}{2}u^2(a^2-c^2)\omega - \tfrac{2}{3}u(a^3-c^3)\omega^2 + \tfrac{1}{4}(a^4-c^4)\omega^3\right\}.$$

From this equation it appears that Wv and ω vary simultaneously; such variations will arise from a variation in the magnitude of W.

Now Wv measures the work performed by the water-wheel: suppose that we propose to find the maximum efficiency of the wheel. Then, equating to zero the differential of Wv with regard to ω, we have

$$0 = \tfrac{1}{2}u^2(a^2-c^2) - \tfrac{4}{3}u(a^3-c^3)\omega + \tfrac{3}{4}(a^4-c^4)\omega^2,$$

whence ω may be found, and then Wv becomes known.

If a be very nearly equal to c, let $c = a - x$, x being a small quantity: then, approximately,

$$0 = \tfrac{1}{2}u^2 \cdot 2xa - \tfrac{4}{3}u \cdot 3a^2x \cdot \omega + \tfrac{3}{4} \cdot 4a^3x \cdot \omega^2,$$

or $$u^2 - 4au\omega + 3a^2\omega^2 = 0,$$

whence $$a\omega = \tfrac{1}{3}u,$$

or the velocity of the float is one-third of that of the current.

Bossut: *Traité d'Hydrodynamique*, tom. I. p. 460.

10. To determine the form of a frustum of a right cone, the base and altitude of which are given, which, moving through a fluid, in the direction of its axis, may experience the least resistance.

Let BD, (fig. 58), be the axis of the frustum, and $ABCD$ the section of the cone made by a plane through BD. Produce AC to meet BD produced in O, which will be the vertex of the complete cone. Take P any point in CA and draw PM at right angles to BD.

Let $AB = c$, $BD = h$, $PM = y$, $CD = y_1$, $\angle AOB = \alpha$, $CP = s$. Then, v denoting the velocity of the frustum, the resistance on its circular end CD will be equal to

$$\tfrac{1}{2}\rho v^2\pi y_1^2,$$

and the resistance arising from the action of the fluid on the convex surface will be equal to

$$\int \tfrac{1}{2}\rho \,.\, 2\pi y ds \,.\, v^2 \sin^3 \alpha$$
$$= \tfrac{1}{2}\rho v^2 \pi \,.\, 2 \sin^3 \alpha \int y ds$$
$$= \tfrac{1}{2}\rho v^2 \pi \,.\, 2 \sin^2 \alpha \int y dy$$
$$= \tfrac{1}{2}\rho v^2 \pi \,.\, \sin^2 \alpha \, (c^2 - y_1^2).$$

Hence the expression

$$u = y_1^2 + \sin^2 \alpha \,.\, (c^2 - y_1^2) = c^2 \sin^2 \alpha + y_1^2 \cos^2 \alpha$$

must be a minimum.

But $$y_1 \cos \alpha = c \cos \alpha - h \sin \alpha,$$

and therefore $u = c^2 - 2ch \sin \alpha \cos \alpha + h^2 \sin^2 \alpha$;

whence, putting $\dfrac{du}{d\alpha} = 0$, we have

$$2h \sin \alpha \cos \alpha - 2c (\cos^2 \alpha - \sin^2 \alpha) = 0,$$
$$c \cot^2 \alpha - h \cot \alpha = c,$$
$$c^2 \cot^2 \alpha - ch \cot \alpha + \tfrac{1}{4}h^2 = c^2 + \tfrac{1}{4}h^2,$$

and therefore, if $x = OB$,

$$x = c \cot \alpha = \tfrac{1}{2}h + (c^2 + \tfrac{1}{4}h^2)^{\frac{1}{2}},$$

which determines the position of the vertex of the complete cone.

This result enables us to give the following construction of the cone. Bisect BD in E, join EA, and, in BD produced indefinitely, set off EO equal to EA. This construction was given without demonstration by Newton in his *Principia*, in the Scholium to Prop. 34, lib. II. sect. 7.

Herman: *Phoronomia*, p. 249.

11. Two opposite edges of a thin inextensible and perfectly flexible rectangular piece of cloth are fixed parallel to each other, and the cloth is exposed to a uniform current of air moving at right angles to the plane which contains the two fixed edges of the cloth: to determine the form of equilibrium of the cloth.

Let the curve KAL (fig. 59) represent a section of the cloth made by a plane at right angles to its two fixed edges, K and L being the two fixed points of the curve. Let Ox, Oy, be rectangular axes of the curve, OAy being its axis of symmetry. The air is supposed to be moving in the direction yO.

Let v denote the velocity of the air: then the unit of resistance p at a point (x, y) of the curve will be equal to

$$\tfrac{1}{2}\rho v^2 \frac{dx^2}{ds^2}.$$

But, t denoting the tension of the curve and r the radius of curvature at this point, $\quad t = pr,$

where t is a constant quantity because the curve is subject only to normal action: hence, a denoting a constant quantity,

$$a = \frac{dx^2}{ds^2} \cdot \frac{\dfrac{ds^3}{dx^3}}{\dfrac{d^2y}{dx^2}},$$

or

$$1 = \frac{a\,\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}};$$

integrating, and observing that x and $\dfrac{dy}{dx}$ are equal to zero simultaneously, we have

$$x = a \log \left\{\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\},$$

whence

$$\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = e^{\frac{x}{a}},$$

and

$$-\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = e^{-\frac{x}{a}},$$

and therefore

$$2\frac{dy}{dx} = e^{\frac{x}{a}} - e^{-\frac{x}{a}},$$

integrating, and supposing the origin to be at a distance a from the vertex, we have

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

This curve is called the Curva Velaria, as being the form of a vertical section of a sail filled by the wind: its equation shews that it coincides in form with the Catenaria or Funicularia.

COR. If we take s as the independent variable, then

$$r = \frac{dx\,ds}{d^2y},$$

and therefore, from the equation $t = pr$, there is

$$a = \frac{dx^2}{ds^2} \cdot \frac{dx\,ds}{d^2y},$$

or

$$ads\,d^2y = dx^3,$$

the form in which the differential equation to the curve was given by James Bernoulli.

The history of the discovery of the Velaria and Lintearia is given by James Bernoulli, in the *Acta Eruditorum* for the year 1695, p. 546, from which the following is an extract:

"Cum ineunte anno 1691 Fratri Genevam misissem proportionem hanc solvendam, $d^2x : dx :: dy^3 : \int dy^3$, qua Velariam comprehendi indicabam, ille ex P. Gastone Pardies retulit, Velum considerari posse instar funiculi pondere carentis, cui infinitæ lineæ æquidistantes et æque graves insistant; adeoque in Prisma Parabolicum curvari, juxta id quod in Actis habetur m. Jun. 1691, p. 288. Sed monitus diversam esse rationem fluidi impellentis, ac solidi ponderis juxta eandem directionem trahentis vel prementis, mox sententiam mutavit, intuensque velum ceu linteum liquore aliquo impletum, indagare cœpit quænam ejus curvatura foret, si pressio fluidi linteo communicaretur secundum directionem horizontalem sive verticalem; harum enim hypothesium alterutram veram esse persuasum habebat: et cum ne hoc quidem probarem, existimabat saltem lintei hujus, si non veli, curvaturam se dedisse; donec ego, paulo post apertius me explicans, hanc fluidorum naturam esse perhiberem, ut pressionem communicent, nec secundum horizontalem, nec verticalem; sed secundum lineam corpori impulso in quovis impulsus puncto perpendicularem; hac tamen cum

differentia, quod ubi fluidum motum suum post impulsum potest prosequi, partem tantum virium in premendo corpore impendat; si vero stagnat alicubi, nec habet quo evadat, omnes suas vires in illud transfundat: hinc velum concipiendum instar funis ab infinitis potentiis æqualibus, aut inæqualibus, tracti vel impulsi, quæ cum sint æquales, manifestum esse formari circulum, quemadmodum etiam per calculum inveneram, eoque in hypotheseos assumptæ veritate prorsus confirmabar. Interea dum ille, sub finem anni 1691, Parisios se confert, transmuto proportionem hanc $d^2x : dx :: dy^3 : \int dy^3$, in æqualitatem $ads\, d^2x = dy^3$, indeque ope methodi cujusdam, quam pro secundis differentiis ad primas reducendis paulo ante excogitaveram, Funiculariam elicio; mox etiam æquationem tecta solutione Fratri Lutetiam mitto, visurus num et sua huc pertingeret.* Is vero rem sibi successisse videns, atque jam factus cupidior sciendi, quomodo ad hanc æquationem pervenerim, denuo ad Veli contemplationem redit, nec cessat, donec animadverterit, artificium in hoc uno consistere, ut singulæ impulsuum directiones in duas alias, horizontalem puta et verticalem resolvantur. Nec mora, protinus inventum prælo committit, ac m. April. 1692, Ephem. Gallicis curat inseri, et quia se solum Problema absolvisse putabat, me de plenaria resolutione desperasse scribit, nescius quod illam jam præcedente Martio una cum Regulis usum inventi concernentibus Lipsiam misissem. Corrigere etiam postea voluit curvaturam suam lintei liquore adimpleti, novamque D. Marchioni Hospitalio solutionem exhibuit, sed eam etiamnum erroneam et a mea diversam. Hanc enim eandem esse cum Elastica, non minus atque Velariam cum Funicularia, constanter sentio; et quod certum veritatis indicium esse potest, identitatem hanc, quam initio ex speciali natura curvarum prolixiore analysi collegi, nunc absque omni fere calculo duabus lineis ostendo."

The deduction of the equation $ads\, d^2x = dy^3$ from the proportion $d^2x : dx :: dy^3 : \int dy^3$ may be effected in the following manner. Since d^2x is to its integral dx as dy^3 to its integral

* The axes of x and y here adopted are the same as those of y and x in the solution of the problem given above.

$\int dy^3$, d^2x will be proportional to dy^3, and therefore, the homogeneity being regarded, we have the equation $adsd^2x = dy^3$: for a and ds are both regarded as constants. Jacobi Bernoulli, *Opera*, tom. I. p. 654, *Note*. For additional information with regard to the Lintearia and Velaria, the reader is referred to the *Opera* Jacobi Bernoulli, tom. I. pp. 481, 576, 639; tom. II. p. 882; and to John Bernoulli's *Lectiones Mathematicæ*, *Lect.* 43, 44; *Opera*, tom. III. p. 510, 512; *Probleme des Isoperimetres*, *Opera*, tom. I. pp. 224, 227, 377, 431; *Manœuvre des Vaisseaux*, *Opera*, tom. II. p. 95.

12. To find the form of a solid of revolution that the resistance in moving through a fluid in the direction of its axis, the resistance being supposed to take place according to the law of the square of the velocity, may be a minimum.*

If the axis of revolution be taken as the axis of x, the form of the generating curve will be defined by the two simultaneous equations

$$x = \frac{3c}{8p^4} + \frac{c}{2p^2} + \frac{c}{2}\log p + c',$$

$$c(1+p^2)^2 = 2yp^3,$$

where c, c', are constants and p represents $\frac{dy}{dx}$. The equation to the curve would result from the elimination, were it possible, of p between these two equations.

Woodhouse: *Isoperimetrical Problems*, p. 115.
Airy: *Mathematical Tracts*, p. 237.

The form of the curve is indicated in the diagram (fig. 60). It consists of two infinite branches terminating in a cusp at B, the tangent at which is inclined at an angle of $\frac{\pi}{3}$ to the axis of revolution.

The conception and solution of this problem are due to Newton, by whom the property of its tangent, as expressed by the equation $$c(1+p^2)^2 = 2yp^3,$$

* This is the earliest problem on record offering an example for the application of the Calculus of Variations.

was enunciated in a geometrical form, although without the demonstration, in his *Principia*, tom. II. p. 269. In the year 1699, an analytical solution of the problem was given by Fatio, at the end of a treatise, published in London, '*De murorum inclinatione fructiferas ad arbores sustinendas benigniorique Soli ostendendas aptissima.*' His analysis, however, was in the highest degree complicated, and in fact he arrived only at the expression for the radius of curvature and second differentials. A copy of Fatio's treatise having been presented by the author to L'Hôpital, this philosopher succeeded in obtaining a much more simple solution of the problem, whence he deduced both the construction of the curve and the property of its tangent discovered by Newton. L'Hôpital's researches on this subject were published in the *Acta Eruditorum* for the year 1699, p. 354, and in the *Mémoires de l'Académie des Sciences de Paris*, for the same year, p. 107. John Bernoulli, dissatisfied as well as L'Hôpital with Fatio's solution, gave one nearly the same as the one by L'Hôpital; (see *Acta Eruditorum*, for the year 1699, p. 514, and for the year 1700, p. 208), and could not refrain from boasting of his facility in doing so: "Hoc enim problema tantæ facilitatis deprehendo, ut ad ejus solutionem nullo prorsus calculo fuerit mihi opus: nam charta et calamo destitutus, et in lecto decumbens, solius imaginationis ope plenarie id solvi." Fatio afterwards resumed his consideration of this problem and published an ingenious memoir entitled "*Solidi rotundi minime resistentis investigatio ex Fermatii doctrina refractionum,*" in the *Acta Eruditorum* for 1701, p. 135. In the year 1713 he inserted in the *Philosophical Transactions* of London a memoir, in which he deduces Newton's property of the tangent from the equation of second differentials, at which he had arrived in the year 1699. The reader may consult also a memoir on this subject by M. De Saint-Jacques De Silvabelle, in the *Mémoires de Mathématique et de Physique Présentés à l'Académie*, tom. III., ann. 1760. Bouguer, in his *Traité du Navire*, liv. III. sect. 5, has considered this very general problem, '*Une base qui est exposée au choc d'un fluide étant donnée, trouver l'espace de conoïde dont*

*il faut la couvrir, pour que l'impulsion soit la moindre qu'il est possible.'**

13. $A'O'O''A''$ (fig. 61) is a vertical and $O'O''B''B'$ a horizontal plane, and $A'O'B'$, $A''O''B''$, are two vertical planes at right angles to the two former: between these four planes there is a quantity of fluid, of which the free surface is $A'A''B''B'$, kept in equilibrium by a current of air flowing at right angles to the plane $A'O'O''A''$: to determine the form of the free surface.

Let $AOBA$ be a section of the fluid made by a plane parallel to $A'O'B'$ or $A''O''B''$. Take P any point in the curve AB and draw PM at right angles to AO. Let $AM = x$, $PM = y$, arc $AP = s$, v = the velocity of the current of air, ρ = the density of the fluid, and σ = the density of the air. Then, for the equilibrium of the fluid, we have

$$dp = \rho g dx,$$
$$p = \rho g x + C,$$

where C is an arbitrary constant.

But, at the point P, the pressure due to the action of the air is equal to

$$\tfrac{1}{2}\sigma\left(v\frac{dx}{ds}\right)^2.$$

Hence, $$\tfrac{1}{2}\sigma v^2\frac{dx^2}{ds^2} = \rho g x + C,$$

or, c denoting a constant,

$$x + c = \frac{\sigma v^2}{2\rho g}\cdot\frac{dx^2}{ds^2}.$$

Putting $\dfrac{\sigma v^2}{2\rho g} = 2a$, we have

$$x + c = 2a\frac{dx^2}{ds^2};$$

whence, by the relation $ds^2 = dx^2 + dy^2$, we get

$$dy = dx\left\{\frac{2a-(x+c)}{x+c}\right\}^{\frac{1}{2}}$$
$$= dx\frac{2a-(x+c)}{\{2a(x+c)-(x+c)^2\}^{\frac{1}{2}}},$$

* *Mémoires de l'Académie des Sciences de Paris*, 1733, p. 85.

the integral of which is

$$y = \{2a(x+c) - (x+c)^2\}^{\frac{1}{2}} + a\int \frac{dx}{\{2a(x+c)-(x+c)^2\}^{\frac{1}{2}}}$$

$$= \{2a(x+c) - (x+c)^2\}^{\frac{1}{2}} + a \text{ vers}^{-1}\frac{x+c}{a} + C',$$

C' being an arbitrary constant: the form of the curve is therefore a cycloid. If A be at the highest point of the curve, $C' = 0$ and $c = 0$, and the equation is

$$y = (2ax - x^2)^{\frac{1}{2}} + a \text{ vers}^{-1}\frac{x}{a}.$$

The radius of the generating circle is given by the equation

$$a = \frac{\sigma v^2}{4\rho g}.$$

Euler has considered this problem also on other hypotheses in regard to the action of fluids incident obliquely.

Euler: *De figura quam ventus fluido stagnanti inducere valet. Acta Acad. Petrop. ann.* 1777, p. 190.

14. A parabolic lamina is placed in a stream flowing parallel to the axis of either of its faces; first with its vertex opposed to the stream, and secondly with its base: to compare the magnitudes of the two resultant pressures.

If P denote the pressure in the former and Q in the latter case, $4m$ be the latus rectum, and y the extreme ordinate of either face,

$$\frac{P}{Q} = \frac{\tan^{-1}\frac{y}{2m}}{\frac{y}{2m}},$$

James Bernoulli: *Acta Eruditorum, Lips.* 1693, p. 253, Art. 4.

Herman: *Phoronomia*, p. 247.

15. A lamina, in the form of a segment of a circle, moves through a fluid in the direction of its axis, first with its vertex and next with its base foremost: to compare the resistances in the two cases.

If R denote the resistance in the former case and R' in the latter, then, D denoting the diameter of the circle and C the length of the chord of the segment,

$$R : R' :: D^2 - \tfrac{1}{3}C^2 : D^2.$$

James Bernoulli: *Act. Erudit. Lips.* 1693, p. 252, Art. 3.
Herman: *Phoronomia*, p. 247.

16. A lamina, in the form of a complete cycloid, moves through a fluid in the direction of its axis, first with its vertex and next with its base foremost: to compare the resistances in the two cases.

The resistance in the former case will be to that in the latter as 3 to 4. Herman: *Phoronomia*, p. 249.

17. To compare the resistance on a sphere placed in a stream with that on a circular plate of the same radius placed at right angles to the stream.

Resistance on the sphere = half that on the plate.

Newton: *Principia*, lib. II. Prop. 34.
Bossut: *Traité d'Hydrodynamique*, tom. I. p. 439.

18. A solid segment of a sphere moves through a fluid in the direction of its axis, first with its vertex and next with its base foremost: to compare the resistances in the two cases.

If R, R', be the resistances in the former and latter case respectively, r the radius of the sphere and y the radius of the base of the segment,

$$R : R' :: r^2 - \tfrac{1}{2}y^2 : r^2.$$

Herman: *Phoronomia*, p. 248.

19. A segment of a paraboloid of revolution moves through a fluid in the direction of its axis, to which its base is perpendicular, first with its vertex and next with its base foremost: to compare the resistances in the two cases.

If y denote the diameter of the base, and l the latus rectum of the generating parabola, the resistance in the former case will be to that in the latter in a ratio denoted by the expression

$$\frac{l^2}{y^2} \log\left(1 + \frac{y^2}{l^2}\right).$$

Herman: *Phoronomia*, p. 248.

20. To determine the resultant resistance on a spheroid moving through a fluid in the direction of its axis.

If $2a$, $2b$, be the axes of the generating ellipse, of which the former is the axis of revolution, and v be the velocity of the spheroid, the required resistance will be equal to

$$\frac{\pi\rho v^2 b^4}{2(a^2 - b^2)} \cdot \left(\frac{2a^2}{a^2 - b^2} \log \frac{a}{b} - 1\right).$$

21. To determine the velocity of the wind, blowing horizontally, when it is just able to overturn a given circular cylinder standing on a horizontal plane and prevented from sliding.

If v represent the velocity and ρ the density of the wind, W the weight and l the length of the cylinder,

$$v^2 = \frac{3W}{\rho l^2}.$$

22. A semicircular arc, of which the radius is a, vibrates about an axis through one extremity and perpendicular to its plane, which is vertical, in a medium where the direct resistance on an element ds of the arc would be equal to $\lambda\rho\,\frac{gvds}{k}$, if v were the normal velocity of ds, λ denoting the area of a transverse section and ρ the density of the arc: to find the length of the isochronous simple pendulum.

The length of the isochronous simple pendulum is equal to

$$\frac{2\pi a}{(\pi^2 + 4)^{\frac{1}{2}} - \dfrac{\pi g a}{32k^2}}.$$

CHAPTER IV.

MOTION OF SOLID BODIES ACTED ON BY FLUIDS.

SECTION I.

Finite Vertical Oscillations in Incompressible Fluids.

IN the solutions of the class of problems which form the subject of this section we shall confine our attention to the motion of floating bodies as subject to the mere statical pressure of the circumambient fluid, without taking account of the modification of the effect of such pressure which arises from the resistance and inertia of the fluid. It will easily be imagined that, such an hypothesis being adopted, the investigation of the motion of floating bodies will be of little value except as affording analytical exercise, and that the results obtained will be entirely at variance with the experiment. The premises which form the basis of this section are chosen merely to simplify the algebraical formulæ which on any true hypothesis would ordinarily be so complicated as to present insuperable difficulties to the analyst.

1. A solid cylinder of given length is pressed down in a vertical position into a fluid, so that its upper end is on a level with the surface, the specific gravity of the cylinder being one-half that of the fluid: the pressure being removed, to find the greatest height to which the upper end of the cylinder will rise above the surface of the fluid.

Let x denote the height of the upper end of the cylinder above the surface of the fluid after any time t from the commencement of the motion; $2a$ the length of the cylinder; κ the

area of its base, ρ the density of the cylinder. Then, for the motion of the cylinder, we have

$$2a\kappa\rho\,\frac{d^2x}{dt^2} = \kappa(2a - x).2g\rho - 2ag\rho\kappa,$$

$$a\,\frac{d^2x}{dt^2} = g(a - x),$$

$$a\,\frac{d^2}{dt^2}(x - a) + g(x - a) = 0.$$

The integral of this equation is

$$x - a = A \sin\left\{\left(\frac{g}{a}\right)^{\frac{1}{2}} t + \beta\right\},$$

A and β being arbitrary constants. But, initially, $x = 0$ and $\frac{dx}{dt} = 0$: hence $\qquad -a = A \sin \beta,$

$$0 = A\left(\frac{g}{a}\right)^{\frac{1}{2}} \cos \beta,$$

and therefore $\qquad x = a\left[1 - \cos\left\{\left(\frac{g}{a}\right)^{\frac{1}{2}} t\right\}\right].$

When the cylinder arrives at its greatest altitude, $\frac{dx}{dt} = 0$:

hence $\qquad 0 = a\left(\frac{g}{a}\right)^{\frac{1}{2}} \sin\left\{\left(\frac{g}{a}\right)^{\frac{1}{2}} t\right\},$

or $\qquad t = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}},$

and $\qquad x = 2a.$

Thus, after a time $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, the cylinder arrives at its greatest altitude, just rising out of the fluid.

2. A solid cylinder, resting in a fluid which is contained in a hollow cylinder with the same vertical axis, is thrust vertically downwards; to ascertain the period of an oscillation.

Let r, r', be the radii of the solid and of the hollow cylinder respectively; ρ the density of the solid cylinder and ρ' of the fluid. Also, let x represent the altitude of the base of the solid

cylinder and x' of the surface of the fluid, above the base of the hollow cylinder, at any time t. Then, for the motion of the oscillating cylinder, there is

$$h\rho \frac{d^2x}{dt^2} = g\rho' (x' - x) - g\rho h \ldots\ldots\ldots\ldots (1).$$

But, c^3 denoting the volume of the fluid,

$$\pi r'^2 x' - \pi r^2 (x' - x) = c^3,$$

$$x' = \frac{c^3 - \pi r^2 x}{\pi (r'^2 - r^2)} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$\frac{d^2x}{dt^2} + \frac{g\rho' r'^2}{\rho h (r'^2 - r^2)} \left\{x - \frac{c^3}{\pi r'^2} + \rho h \frac{r'^2 - r^2}{\rho' r'^2}\right\} = 0,$$

and therefore the period of oscillation is equal to

$$\frac{\pi}{r'} \cdot \left\{\frac{\rho h (r'^2 - r^2)}{g\rho'}\right\}^{\frac{1}{2}}.$$

3. A solid, the surface of which is generated by the revolution of the curve

$$y \propto x^{\frac{n}{2} - 1}$$

round the axis of x, floats in a fluid with a portion h of its axis immersed: to find how much the solid must be depressed, that it may, on its return, just emerge from the fluid.

The required depth of depression is equal to

$$(n^{\frac{1}{n-1}} - 1)\, h.$$

4. A sphere, of specific gravity, ρ is immersed in a fluid of specific gravity ρ', the depth of the centre of the sphere below the surface of the fluid being equal to n times its radius: supposing the sphere to be allowed to ascend, to find the values of n, first, that it may rise just half out of the fluid, and, secondly, that it may rise just entirely out.

The two required values of n are

$$\frac{3\rho'}{16(\rho' - \rho)} \quad \text{and} \quad \frac{\rho}{\rho' - \rho}.$$

5. A hollow cylinder, closed at one end, is depressed vertically from the atmosphere into a fluid, the open end downwards, so that the closed end is in the surface of the fluid; the pressure being then removed, and the height (h) of the cylinder being so small, compared with the height (h') of a column of the fluid the pressure of which is equal to that of the atmosphere, that $\left(\frac{h}{h'}\right)^2$ may be neglected: to find the position of the cylinder when it has risen to its greatest altitude.

If m denote the mass of the cylinder, m' of a volume of the fluid equal in bulk to that of the cylinder, and x the depth of the lower end of the cylinder below the surface just as the cylinder arrives at its greatest altitude,

$$x = \frac{mh}{m'h'} \cdot (h + h').$$

6. Two cylindrical weights with vertical axes, which are connected together by a fine inextensible string passing over a pully without inertia, are in equilibrium, a portion of one of them being immersed in a vessel full of fluid; supposing the surface of the fluid to sink uniformly, to determine the period of the oscillatory motion assumed by the weights relatively to the surface of the fluid.

If P be the weight, σ the density, and h the length of the cylinder which is partially immersed in the fluid, Q the weight of the other cylinder, and ρ the density of the fluid, the required time will be equal to

$$\pi \left\{\frac{\sigma h (P + Q)}{\rho g P}\right\}^{\frac{1}{2}}.$$

SECTION II.

Motion of bodies subject to the action of Aeriform Fluids.

1. A heavy piston descends by its own weight in a smooth closed cylinder, into which it just fits and which is filled with

atmospheric air, the axis of either cylinder being vertical: to determine the velocity of the piston in any position, its lower end being supposed to be initially in contact with the upper extremity of the cylinder.

Let m denote the mass of the piston, p the pressure of atmospheric air on a unit of area, a the radius of the cylinder or piston, v the velocity of the piston when its lower end is at a distance x from the bottom of the cylinder; h the initial value of x, when the air in the cylinder is in its ordinary state of density. Then, for the motion of the piston, observing that $p\,\dfrac{h}{x}$ is the value of the elasticity of the air corresponding to any position of the piston, we have

$$mv\,\frac{dv}{dx} = \pi a^2 p\,\frac{h}{x} - mg,$$

whence, integrating, we obtain

$$mv^2 = C + 2\pi p a^2 h \log x - 2mgx,$$

C being an arbitrary constant. Now, v is equal to zero when x is equal to h, and therefore

$$0 = C + 2\pi p a^2 h \log h - 2mgh;$$

hence $$v^2 = \frac{2\pi p a^2 h}{m}\log\frac{x}{h} + 2g\,(h - x),$$

which determines the velocity required.

2. A piston of given mass is placed at a certain distance from one end of a horizontal cylinder, and the space between that end and the piston is suddenly filled with sufficient steam to produce a given pressure upon the end of the piston as soon as the steam is completely enclosed: to find the velocity of the piston at any distance from its original position, the friction being neglected, and to ascertain what velocity would be destroyed by a very slight friction.

Let m denote the mass of the piston, a its initial distance from the end of the cylinder in contact with the steam, P the initial pressure of the steam on the whole end of the piston. The pressure of the steam upon the end of the piston when

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altitude to which the lower end of the piston will ascend from the bottom, will be defined by the equation

$$2 \log \frac{x}{a} = \frac{x}{a} - 1.$$

4. A balloon starts from the surface of the earth: if D represent the density of the air at the surface of the earth, and B the mean density of the balloon, to determine the motion of the balloon, B being nearly equal to D, and k denoting the constant ratio of the pressure to the density of the atmosphere.

If x denote the number of feet in the altitude of the balloon, at the end of t seconds, above the surface of the earth,

$$x = \frac{k}{g} \frac{D - B}{D} \cdot \left[1 - \cos\left\{g\left(\frac{D}{Bk}\right)^{\frac{1}{2}} t\right\}\right],$$

whence we see that the balloon will oscillate through a vertical space $\frac{2k}{g}\left(1 - \frac{B}{D}\right)$ feet, and perform the oscillations in $\frac{2\pi}{g}\left(\frac{kB}{D}\right)^{\frac{1}{2}}$ seconds.

Section III.

Small Oscillations of Floating Bodies.

The principal questions relative to the oscillations of bodies floating on fluids were investigated successively and at different times by Daniel Bernoulli,* Euler,† D'Alembert,‡ and Bossut.‖ The solution of the general problem of the coexistence of the angular and vertical oscillations of a floating body, symmetrical with regard to a vertical plane, which we have given in this section, is due to Mr. Greathead, by whom it was inserted in the *Cambridge Mathematical Journal,* for November,

* *Commentarii Academiæ Scientiarum Petropolitanæ*, 1739, p. 100.

† *Scientia Navalis seu Tractatus de Construendis ac dirigendis Navibus*, 1749.

‡ *Essai d'une Nouvelle théorie de la Résistance des fluides*, chap. VI. 1752; and *Opuscules Mathématiques*, tom. I.; *Troisième Mémoire :* 1761.

‖ *Sur l'arrimage des Vaisseaux, Prix de l'Académie des Sciences de Paris*, 1761, 1765.

1838. For the investigation of the laws of the small oscillations of a floating body, under the most unrestricted circumstances of form and initial disturbance, the reader is referred to a Memoir by M. Molins, published in Liouville's *Journal de Mathématiques, Tome troisième*, année 1838.

1. A straight thin rod AB, (fig. 62), of uniform thickness, consists of two equal lengths AC, CB: the density of the lower portion BC is double of that of the higher portion AC: supposing this rod to be almost totally immersed, at a slight inclination to the vertical, in a fluid of which the density is just greater than three halves of that of the upper portion of the rod, to find the length of a simple pendulum which will vibrate isochronously with the angular oscillations of the rod.

Let C be the centre and G the centre of gravity of AB. Let $AB = a$, $CG = b$, $\theta =$ the inclination of AB to the vertical, $M =$ the mass of the rod, $k =$ its radius of gyration about G, $\kappa =$ the area of a section of the rod, $\rho =$ the density of AC.

Then the action of the fluid on the rod will be equivalent to a vertical force through C nearly equal to $\frac{3}{2}a\kappa\rho g$, and therefore, for the motion about G,

$$Mk^2\frac{d^2\theta}{dt^2} = -b\sin\theta \cdot \tfrac{3}{2}a\kappa\rho g,$$
$$= -\tfrac{3}{2}ab\kappa\rho g\theta, \quad \text{nearly.}$$

But, if we suppose every element of AC and half of every element of BC to be condensed at C, it is plain that the centre of gravity of AB will not be affected; hence

$$b = \frac{\rho \cdot \frac{1}{2}a\kappa \cdot \frac{1}{4}a}{(\rho + 2\rho) \cdot \frac{1}{2}a\kappa} = \frac{a}{12}.$$

We have, therefore,

$$Mk^2\frac{d^2\theta}{dt^2} = -\tfrac{1}{8}a^2\kappa\rho g\theta.$$

Let k_1 denote the radius of gyration of the rod AB about C; then

$$Mk_1^2 = \kappa(\rho + 2\rho)\int_0^{\frac{1}{2}a} r^2 dr$$
$$= 3\kappa\rho \cdot \tfrac{1}{3}\left(\frac{a}{2}\right)^3 = \tfrac{1}{8}\kappa\rho a^3,$$

and therefore
$$Mk^2 = \tfrac{1}{8}\kappa\rho a^3 - M \cdot \frac{a^2}{(12)^2}$$
$$= \kappa\rho\left\{\frac{a^3}{8} - \frac{a}{2}(\rho + 2\rho)\frac{a^2}{(12)^2}\right\}$$
$$= \kappa\rho a^3\left\{\frac{1}{8} - \frac{3}{2(12)^2}\right\}$$
$$= \tfrac{11}{96}\kappa\rho a^3.$$

The equation of motion becomes, therefore,

$$\tfrac{11}{96}\kappa\rho a^3 \frac{d^2\theta}{dt^2} = -\tfrac{1}{8}a^2\kappa\rho g\theta,$$
$$\frac{d^2\theta}{dt^2} + \frac{12g}{11a}\theta = 0.$$

This result shews that the length of the required pendulum will be equal to $\tfrac{11}{12}a$.

Daniel Bernoulli: *De Motibus oscillatoriis corporum humido insidentium; Comment. Acad. Petrop.* 1739, p. 106.

2. A body, which is symmetrical with regard to a vertical section, executes small oscillations in a fluid in such a manner that the motions of all the particles of the body take place parallel to the vertical section: to determine the nature of the vertical and angular oscillations of the body, the centre of gravity of the plane of floatation, when the body is at rest, not lying in the vertical line through the centres of gravity of the body and the fluid displaced.

Let AB (fig. 63) be the projection, on the plane of symmetry, of the section of the body which, when the body is in equilibrium, constitutes its plane of floatation: let C be the centre of gravity of the plane AB, ab the projection of the plane of floatation at any time t. Draw $a'b'$ parallel to ab, through C. Let G be the centre of gravity of the body, H that point of the body which, when the body is at rest, coincides with the centre of gravity of the fluid displaced. Let x be the vertical distance of the centre of gravity of the body below its place of rest at the

time t, θ the angular deviation from the position of equilibrium. Then the equations of motion are

$$M\frac{d^2x}{dt^2} = (M - M')\,g \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$Mk^2\frac{d^2\theta}{dt^2} = M'y \cdot g \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

M being the mass of the body, M' that of the fluid displaced, viz. aDb, y the horizontal distance of the centre of gravity of the fluid displaced from G at the time t, parallel to ba, Mk^2 the moment of inertia of the body about a line through G at right angles to the section of symmetry.

We must express M' and $M'y$ approximately in terms of x and θ. Produce HG to h: the line HGh will of course be at right angles to AB: draw Ck vertically. Let K = the area of the section AB, $a = Ch$, $\alpha = \angle CGh$, ρ = the density of the fluid.

Then the distance of G below the surface

$$= Ck + CG \sin GCb' = Ck + CG\cos(\alpha - \theta)$$

$$= Ck + CG(\cos\alpha + \theta\sin\alpha), \text{ nearly},$$

$$= Ck + CG\cos\alpha + a\theta:$$

but, when the body is floating at rest, the depth of G is equal to $CG\cos\alpha$: hence, x being the difference between these two depths,

$$x = Ck + a\theta.$$

Again, λ denoting a small area at a point P in the section $a'Cb'$, z the length of a perpendicular upon it, cut off by the section ACB, and x_1 the distance of λ from a line through C, at right angles to the plane of symmetry, we have

$$M' = \text{mass of } ADB - \rho\Sigma(\lambda z) + \text{mass of } aa'b'b,$$

Σ denoting summation for the two wedges ACa', BCb', in the former of which z is positive, in the latter negative; or

$$M' = M - \rho\theta\Sigma(\lambda x_1) + \rho K \,.\, Ck, \text{ nearly},$$

$$= M - \rho\theta\Sigma(\lambda x_1) + \rho K(x - a\theta):$$

but, C being the centre of gravity of the area ACB, and therefore approximately of the area $a'Cb'$, $\Sigma(\lambda x_1) = 0$; hence

$$M' = M + \rho K(x - a\theta).$$

We have therefore, from (1),

$$M\frac{d^2x}{dt^2} = -g\rho K(x - a\theta) \ldots\ldots\ldots\ldots(3).$$

Again, the action of the fluid aDb on the body is equivalent to that of $aa'b'b$ together with that of $a'Db'$.

Now the moment of $aa'b'b$ about G

$$= g\rho K . Ck . CG \cos GCb', \quad \text{nearly},$$
$$= g\rho K . (x - a\theta) . a, \quad \text{nearly},$$

and tends to twist the body in the direction of the arrows in the figure. Also, the moment of the fluid $a'Db'$ about G, if $HG = b$, and c = the projection of CG on Cb', is equal to

$$Mgb\theta - g\rho\Sigma . \lambda z(x_1 + c),$$

in the direction of the arrows, the Σ being supposed to extend to the two wedges ACa', BCb', in the former of which, z and x' are positive, in the latter, negative; or the moment of the fluid $a'Db'$ is equal to

$$Mgb\theta - g\rho\theta\Sigma\lambda x_1(x_1 + c), \quad \text{nearly},$$
$$= Mgb\theta - g\rho\theta\Sigma\lambda x_1^2 - cg\rho\theta\Sigma(\lambda x_1),$$

and therefore, $\Sigma(\lambda x_1)$ being nearly zero, since the centre of gravity of the area ACB and therefore aproximately of the area $a'Cb'$ is at C, we see that the moment of $a'Db'$

$$= g\theta\{Mb - \rho\Sigma(\lambda x_1^2)\} = g\theta(Mb - \rho I),$$

where I denotes the moment of inertia of the area $a'b'$, or, which is approximately the same, of the area ACB about the line through C which cuts the section of symmetry at right angles.

Thus $\quad M'y . g = g\rho K(x - a\theta) . a + g\theta(Mb - \rho I).$

We have therefore, from (3) and (2),

$$\frac{d^2x}{dt^2} + \frac{g\rho K}{M}(x - a\theta) = 0,$$

$$\frac{d^2\theta}{dt^2} = \frac{\rho Kag}{Mk^2}x - \frac{g}{Mk^2}\{\rho(I + Ka^2) - Mb\} . \theta;$$

which are two linear differential equations which may easily be integrated.

Assume, for the sake of simplicity,

$$\frac{g\rho K}{M} = m^2, \quad \frac{\rho(I + Ka^2) - Mb}{\rho K a} = f;$$

then
$$\frac{d^2x}{dt^2} + m^2(x - a\theta) = 0,$$

$$\frac{d^2\theta}{dt^2} - \frac{m^2 a}{k^2}(x - f\theta) = 0.$$

Assume $\quad x = A \sin(\mu t + \epsilon), \quad \theta = \beta \sin(\mu t + \epsilon)$:

then $\; -A\mu^2 + m^2(A - a\beta) = 0, \quad$ or $\; A(m^2 - \mu^2) = a\beta m^2$;

and
$$-\beta\mu^2 - \frac{m^2 a}{k^2}(A - f\beta) = 0,$$

or
$$\beta\left(\frac{m^2 af}{k^2} - \mu^2\right) = \frac{m^2 a}{k^2} A:$$

whence, eliminating A and β,

$$(m^2 - \mu^2)\left(\frac{m^2 af}{k^2} - \mu^2\right) = \frac{m^4 a^2}{k^2}.$$

Let μ_1^2, μ_2^2, be the two values of μ^2 deducible from this equation: then, A_1, A_2, β_1, β_2, ϵ_1, ϵ_2, being arbitrary constants,

$$x = A_1 \sin(\mu_1 t + \epsilon_1) + A_2 \sin(\mu_2 t + \epsilon_2),$$
$$\theta = \beta_1 \sin(\mu_1 t + \epsilon_1) + \beta_2 \sin(\mu_2 t + \epsilon_2),$$

by which equations the motion is completely determined.

It must be observed that

$$A_1 = \frac{m^2 af - \mu_1^2 k^2}{m^2 a}\beta_1, \quad A_2 = \frac{m^2 af - \mu_2^2 k^2}{m^2 a}\beta_2.$$

Cor. Suppose that the centre of gravity of the plane of floatation, when the body is at rest, lies in the vertical line through G and H. Then, putting $a = 0$, the two differential equations degenerate into

$$\frac{d^2x}{dt^2} + \frac{g\rho K}{M} x = 0,$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{Mk^2}(\rho I - Mb)\,\theta = 0,$$

the integrals of which are

$$x = A \sin\left\{\left(\frac{g\rho K}{M}\right)^{\frac{1}{2}} t + \varepsilon\right\},$$

$$\theta = \beta \sin\left\{\left(\frac{g}{Mk^2}\right)^{\frac{1}{2}} (\rho I - Mb)^{\frac{1}{2}}\, t + \zeta\right\},$$

the period of the vertical and angular oscillations being respectively

$$\pi\left(\frac{M}{g\rho K}\right)^{\frac{1}{2}}, \quad \frac{\pi (Mk^2)^{\frac{1}{2}}}{g^{\frac{1}{2}}(\rho I - Mb)^{\frac{1}{2}}}.$$

The amplitudes A and β of the oscillations are independent of each other, and their periods are different. Such, it may be observed, is not the case under the more general form of the problem, where x and θ both involve two terms of periods $\frac{\pi}{\mu_1}$ and $\frac{\pi}{\mu_2}$, the amplitudes A_1, β_1, and A_2, β_2, being not independent of each other.

3. A paraboloid of revolution, with its axis vertical and vertex downwards, is oscillating in a fluid; having given the period of its small oscillations, to determine the depth of its immersion when it is in a position of rest.

If V denote the volume of the fluid displaced by the paraboloid in a position of rest, K the area of its plane of floatation, and x the distance of the centre of gravity of the paraboloid below its place of rest at the end of any time t; then, observing that M is equal to ρV, we have

$$\frac{d^2x}{dt^2} + \frac{gK}{V} x = 0.$$

Let a represent the length of the axis immersed, when the body is at rest: then, the volume of a paraboloid of revolution being equal to one-half of its circumscribing cylinder,

$$V = \tfrac{1}{2} Ka,$$

and therefore

$$\frac{d^2x}{dt^2} + \frac{2g}{a} x = 0,$$

$$x = A \sin\left\{\left(\frac{2g}{a}\right)^{\frac{1}{2}} t + \varepsilon\right\},$$

A, ε, being arbitrary constants.

If T denote the time of an oscillation,

$$T = \pi \left(\frac{a}{2g}\right)^{\frac{1}{2}},$$

$$a = \frac{2gT^2}{\pi^2}.$$

4. An isosceles triangle floats in a fluid with its vertex downwards; supposing the triangle to experience a slight angular displacement about a line perpendicular to its plane, to find the period of its small angular oscillations.

Let ABC (fig. 64) be the triangle, with its axis CD vertical; let G, H, be the centres of gravity of the triangle and of the fluid displaced respectively.

The time of oscillation is equal to

$$\frac{\pi k}{g^{\frac{1}{2}}\left(\frac{I}{V} - b\right)^{\frac{1}{2}}},$$

where k = the radius of gyration of the triangle about a normal line through G, I = the moment of inertia of the line of floatation about a line through D at right angles to the plane of the triangle, $b = GH$, and V = the volume of the fluid displaced. Let $AD = c = BD$, $CD = h$.

Let I' denote the moment of inertia of the triangle about an axis through C at right angles to its plane; then

$$I' = ch.k^2 + ch.(\tfrac{2}{3}h)^2 = ch(k^2 + \tfrac{4}{9}h^2) \ldots\ldots\ldots\ldots(1).$$

Now $$I' = \int_0^h\int_{-y'}^{+y'}(x^2 + y^2)\,dxdy = \int_0^h(2x^2y' + \tfrac{2}{3}y'^3)\,dx,$$

where $y' = \dfrac{cx}{h}$; and therefore

$$I' = \int^h\left(2x^2.\frac{cx}{h} + \frac{2}{3}\frac{c^3x^3}{h^3}\right)dx$$

$$= \frac{2c}{h}\int_0^h\left(1 + \frac{c^2}{3h^2}\right)x^3dx$$

$$= \frac{2c}{3h^3}(3h^2 + c^2).\frac{h^4}{4} = \tfrac{1}{6}ch(3h^2 + c^2).$$

Hence, from (1), we have

$$k^2 = \tfrac{1}{6}(3h^2 + c^2) - \tfrac{4}{9}h^2 = \tfrac{1}{18}(3c^2 + h^2) \dots\dots (2).$$

Let c' = half the line of floatation: then we shall have

$$I = \int_{-c'}^{c'} y^2 dy = \tfrac{2}{3}c'^3 \dots\dots (3).$$

Again, ρ denoting the density of the triangle and ρ' of the fluid,

$$V\rho' = ch\rho, \quad V = \frac{ch\rho}{\rho'} \dots\dots (4).$$

But, by similar triangles, it is plain that

$$\frac{c'^2}{c^2} = \frac{V}{ch} = \frac{\rho}{\rho'}, \quad \text{by (4)},$$

and therefore $$c' = \left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}} c:$$

hence, from (3), $$I = \tfrac{2}{3}\left(\frac{\rho}{\rho'}\right)^{\frac{3}{2}} c^3 \dots\dots (5).$$

Again, by similar triangles,

$$CH^2 = CG^2 \times \frac{V}{ch} = (\tfrac{2}{3}h)^2 . \frac{\rho}{\rho'} = \frac{4\rho}{9\rho'} h^2;$$

whence $$CH = \frac{2}{3}\left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}} . h,$$

and therefore $$b\,(= HG) = CG - CH = \tfrac{2}{3}h\left\{1 - \left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}}\right\} \dots (6).$$

Hence, substituting in the formula for the time of oscillation, the values of k, I, V, b, given in (2), (5), (4), (6), we get

$$\text{time of oscillation} = \frac{\pi(3c^2 + h^2)^{\frac{1}{2}}}{(12g)^{\frac{1}{2}}\left[\left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}}\frac{c^2}{h} - h\left\{1 - \left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}}}.$$

Encycl. Metrop. Mixed Sc., vol. I. p. 194.

5. A square lamina of uniform density and thickness is floating on a fluid with two sides vertical: supposing the lamina to experience a slight angular displacement about a line perpendicular to its plane, to find the length of a simple

pendulum which shall vibrate isochronously with the consequent angular oscillations.

If a represent the length of a side of the lamina, ρ the density of the fluid, and ρ' of the lamina; then, L denoting the length of the required pendulum,

$$L = \frac{2a\rho\rho'}{\rho\rho' - 6\rho\rho' + 6\rho'^2}.$$

Daniel Bernoulli: *De Motibus oscillatoriis corporum humido insidentium; Comment. Acad. Petrop.* 1739, p. 106.

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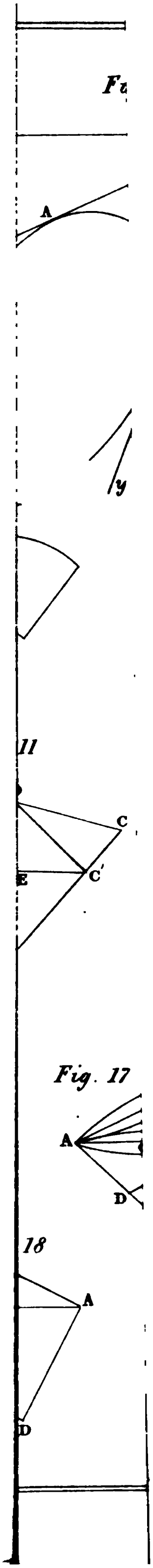

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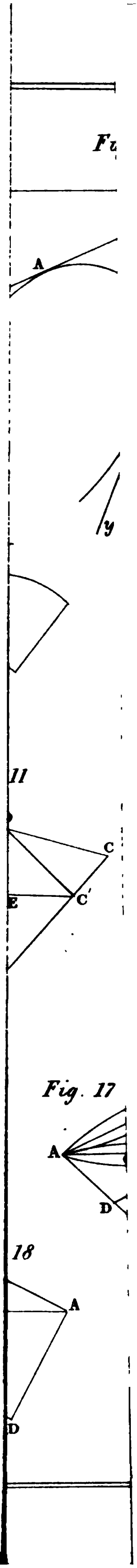
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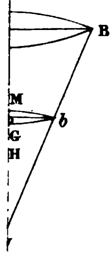

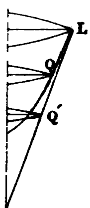

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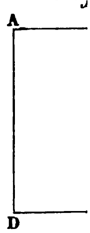

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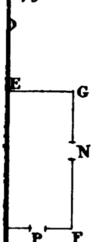

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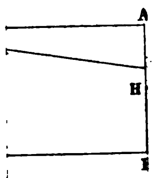

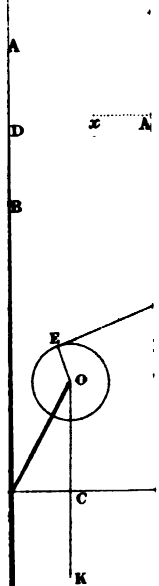

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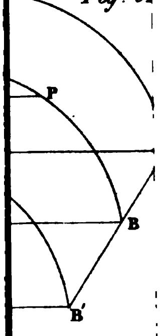

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